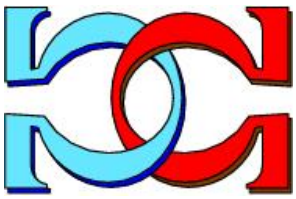

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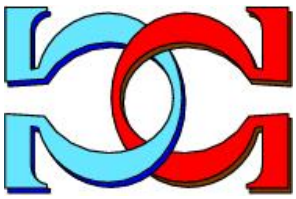
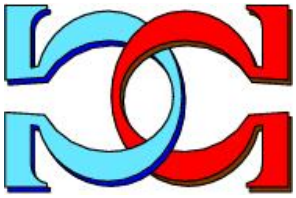


P Systems and the Byzantine Agreement



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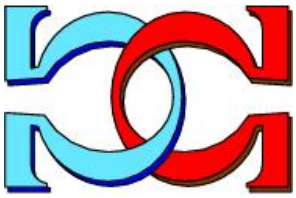
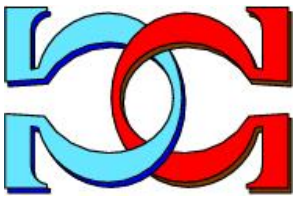
Yun-Bum Kim
Michael J. Dinneen



Department of Computer Science,
University of Auckland,
Auckland, New Zealand



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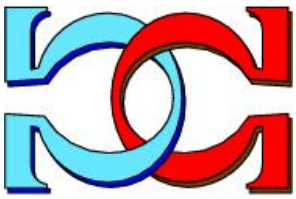

Centre for Discrete Mathematics and
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P Systems and the Byzantine Agreement

Michael J. Dinneen, Yun-Bum Kim and Radu Nicolescu
Department of Computer Science, University of Auckland,
Private Bag 92019, Auckland, New Zealand

Abstract

We first propose a modular framework for recursive composition of P systems. This modular approach provides encapsulation and information hiding, facilitating the design of P programs for complex algorithms. Using this framework, we developed a P program that solves the classical version of the Byzantine agreement problem, for N participants connected in a complete graph, according to the well known Byzantine agreement algorithm based on EIG trees. We prove the correctness of this modular composition and conclude with a list of open problems.

1 Introduction

This paper continues our study of P systems [Pau02, Pău06] as modelling tools for distributed applications and networking, initially motivated by the investigations of Ciobanu *et al.* [CDK02, Cio03]. We earlier proposed a new model for P systems, called *hyperdag P systems* [NDK08, NDK09b], in short *hP systems*, which allows more flexible communications than tree-based models, while preserving a strong hierarchical structure. To achieve our goals, this model has subsequently evolved [NDK09c, NDK09a, DKN09] and it offers the following distinct facilities: (a) it extends the tree structure of classical P systems to *directed acyclic graphs* (dags); (b) it augments the operational rules of neural P systems (nP systems) [Pau02] with *broadcast* facilities; (c) it refines the *rewriting* and *transfer* modes, associating these modes independently to each rule, instead of state; and (d) it allows the creation of *mobile* channels, which dynamically extend the structure of a considered P system model (analogous to nerves which extend in a regenerating tissue or threads extended by spiders). We have noticed that these adjustments, which enhance the model versatility, can also be retrofitted to other P system models.

Using this model, we developed basic building blocks in [NDK09a], that are relevant for network discovery (see also [Lyn96]): broadcast, convergecast, flooding, determine shortest paths and other basic metrics (such as, the number of nodes, descendants, paths).

We also studied the well known *Firing Squad Synchronization Problem* (FSSP), in the framework of P systems [DKN09]. We provided efficient solutions for the FSSP problem that have wider applicability than previous solutions [BGMV08, AMV08].

Here, we continue this study to address the possible existence of cells that are arbitrarily faulty. A well-studied problem in this area is known as the *Byzantine agreement*

problem, first proposed in 1980 [PSL80]: reliable computer systems (or networks) must be able to handle malfunctioning components (or processes) that give conflicting information to different parts of the system. Lamport *et al.*'s description [LSP82] is very readable and this problem has become one of the most studied problems in distributed computing—some even consider it the "crown jewel" of distributed computing. Lynch covers many versions of this problem and their solutions, including a complete description of the classical algorithm based on EIG trees [Lyn96].

Recent years have seen revived interest in this problem and its solutions, in a wide variety of contexts [CL02, AEMGG$^+$05, CKS05, MA06], including, for example, solutions for quantum computers [BOH05, Wik09b]. To the best of our knowledge, no solution for P systems has been published. In the context of P systems, this problem was briefly mentioned, without solutions [CDK02, Cio03]. We believe that we provide the first P systems solution for this problem. Our solution is based on the classical algorithm, using EIG trees.

In the course of this work, we realized that our framework was not versatile enough for a reasonable design. Following Paun *et al.*'s proposal [PPJ10], we propose a new modular framework, which supports encapsulation, information hiding and recursive composition. Our proposal is compatible with any data structure based on directed arcs, i.e., it covers cell-like P systems (based on trees), hP systems (based on dags) and nP systems (based on digraphs).

The rest of the paper is organized as follows. Section 2 covers a few basic preliminaries, then introduces our new modular framework, called P modules, and the recursive composition of P modules. We describe the Byzantine agreement problem in detail in Section 3, which also includes a small case study with four processes. Section 4 introduces the classical Byzantine agreement algorithm based on EIG trees. In Section 5, using our new modular framework, we model and develop the structure of a P systems implementation of the Byzantine agreement problem. The rules used in our design are described in Section 6. In Section 7, we prove the correctness of our modular design. Finally, in Section 8, we summarize our results and discuss related open problems.

2 Preliminaries

We assume that the reader is familiar with the basic terminology and notations: relations, graphs, nodes (vertices), arcs, directed graphs, dags, trees, alphabets, strings and multisets [NDK08].

We first recall a few basic concepts from combinatorial enumerations. The *integer range* from m to n is denoted by $[m, n]$, i.e., $[m, n] = \{m, m + 1, \ldots, n\}$, if $m \leq n$, and $[m, n] = \emptyset$, if $m > n$. The set of *permutations* of n of length m is denoted by $P(n, m)$, i.e., $P(n, m) = \{\pi : [1, m] \to [1, n] \mid \pi \text{ is injective}\}$. A permutation π is represented by the sequence of its values, i.e., $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$, and we will often abbreviate this further as the sequence $\pi = \pi_1.\pi_2 \ldots \pi_m$. The sole element of $P(n, 0)$ is denoted by $()$. Given a subrange $[p, q]$ of $[1, m]$, we define a *subpermutation* $\pi(p : q) \in P(N, q - p + 1)$ by $\pi(p : q) = (\pi_p, \pi_{p+1}, \ldots, \pi_q)$. The *image* of a permutation π, denoted by $Im(\pi)$, is the set of its values, i.e., $Im(\pi) = \{\pi_1, \pi_2, \ldots, \pi_m\}$. The *concatenation* of two permutations

is denoted by $\oplus$, i.e., given $\pi \in P(n,m)$ and $\tau \in P(n,k)$, such that $Im(\pi) \cap Im(\tau) = \emptyset$, $\pi \oplus \tau = (\pi_1, \pi_2, \ldots, \pi_m, \tau_1, \tau_2, \ldots, \tau_k) \in P(n, m+k)$.

The Byzantine agreement algorithm used later in this paper uses *Exponential Information Gathering (EIG)* trees as a data structure. An EIG tree $T_{N,L}$, $N \geq L \geq 1$, is a labelled (ordered) rooted tree of height L that is defined recursively as follows. The tree $T_{N,1}$ is a rooted tree with $1+N$ nodes, with root labelled by λ and its N leaves labelled $1, 2, \ldots, N$, left to right. For $L > 1$, $T_{N,L}$ is a rooted tree with $1+N|T_{N-1,L-1}|$ nodes, root λ, having N subtrees, where each subtree is isomorphic with $T_{N-1,L-1}$ and each subtree node is labeled by the least element of $[1, N]$ that is different from any ancestor node or any left sibling node. Thus, there is a bijective correspondence between the permutations of $P(N, L)$ and the sequences (concatenations) of labels on all root-to-leaf paths of $T_{N,L}$. See Figure 2 for an example of the EIG tree $T_{4,2}$.

We also assume familiarity with P systems, nP systems or hP systems. Although the P systems considered here can be described in these classical frameworks, we prefer to present them in a modular way, using a new definition, that subsumes their essential features and provides facilities for recursive modular composition.

Definition 1 (P module)**.** A *P module* is a system $\Pi = (O, K, \delta, P)$, where:

1. O is a finite non-empty alphabet of *objects*;
2. K is a finite set of *cells*, where each cell, $\sigma \in K$, has the form $\sigma = (Q, s_0, w_0, R)$, where:
   - Q is a finite set of *states*;
   - $s_0 \in Q$ is the *initial state*;
   - $w_0 \in O^*$ is the *initial multiset* of objects;
   - R is a finite *ordered* set of multiset rewriting *rules* of the general form: $s\ x \rightarrow_\alpha s'\ x'\ (u)_{\beta_\gamma}$, where $s, s' \in Q$, $x, x' \in O^*$, $u \in O^*$, $\alpha \in \{min, max\}$, $\beta \in \{\uparrow, \downarrow, \updownarrow\}$, $\gamma \in \{one, spread, repl\} \cup K$. If $u = \lambda$, this rule can be abbreviated as $s\ x \rightarrow_\alpha s'\ x'$. The meaning of operators α, β, γ is described at the end of this definition.
3. δ is a binary relation on K, i.e., a set of parent–child arcs, representing *duplex* or *simplex* channels between cells;
4. P is a subset of K, indicating the *port* cells, i.e., the only cells can be connected to other modules.

The rules given by the ordered set R are attempted in *weak priority* order [Pău06]. If a rule is *applicable*, then it is *applied* and then the next rule is attempted (if any). If a rule is not applicable, then the next rule is attempted (if any). Note that state-based rules introduce an extra requirement for determining rule applicability, namely the target state indicated on the right-hand side must be the same as the previously chosen target state (if any) [Pau02, NDK08, NDK09b]. Rules are applied under the usual eager evaluation of their left-hand sides and lazy evaluation of their right-hand sides [Pau02].

With these conventions, one cell's ordered set of rules becomes a sequence of programming statements for a hypothetical P machine, where each rule includes a simple if-then-fi conditional test for applicability and, as we see below, some while-do-od looping facilities (*max* and *repl* operators), with some potential for in-cell parallelism, in addition to the more obvious inter-cell parallelism. State compatibility introduces another intra-cell if-then-fi conditional test, this time between rules.

The *rewriting* operator $\alpha = min$ indicates that the rewriting is applied once, if the rule is applicable; and $\alpha = max$ indicates that the rewriting is applied as many times as possible, if the rule is applicable. Here, we intentionally avoid the $\alpha = par$ operator, because we do not use it and it is more complicated to integrate it into a priority scheme.

The *transfer* operator $\beta = \uparrow$ indicates that the multiset u is sent "up" to the parents; $\beta = \downarrow$ indicates that the multiset u is sent "down" to the children; and $\beta = \updownarrow$ indicates that the multiset u is sent both "up" and "down". For simplicity, here we intentionally avoid other operators that we do not use in this paper, such as $\beta = \leftrightarrow$, which indicates transfer to the siblings.

The additional transfer operator $\gamma = one$ indicates that the multiset u is sent to one recipient (parent or child, according to the direction indicated by β). The operator $\gamma = spread$ indicates that the multiset u is spread among an arbitrary number of recipients (parents, children or parents and children, according to the direction indicated by β). The operator $\gamma = repl$ indicates that the multiset u is replicated and broadcast to all recipients (parents, children or parents and children, according to the direction indicated by β). The operator $\gamma = \sigma \in K$ indicates that the multiset u is sent to σ, if cell σ is in the direction indicated by β; otherwise, the multiset u is "lost". By convention, if cells have unique indices or are labelled and labels are locally unique, we can abbreviate $\gamma = \sigma$ by $\gamma = i$, where i is the index or label of σ.

The following examples illustrate the behaviour of these operators. Consider a cell σ, in state s and containing aa. Consider the potential application of a rule $s\ a \rightarrow_{\alpha} s'\ b\ (c)_{\beta_{\gamma}}$, by looking at specific values for α, β, γ operators:

- The rule $s\ a \rightarrow_{min} s'\ b\ (c)_{\uparrow_{repl}}$ can be applied and, after its application, cell σ will contain ab and a copy of c will be sent to each of σ's parents.

- The rule $s\ a \rightarrow_{max} s'\ b\ (c)_{\uparrow_{repl}}$ can be applied and, after being applied twice, cell σ will contain bb and a copy of cc will be sent to each of σ's parents.

- The rule $s\ a \rightarrow_{min} s'\ b\ (c)_{\downarrow_{\sigma'}}$ (where $\sigma' \in K$), can be applied and, after its application, cell σ will contain ab and a copy of c will be sent to σ', if σ' appears among the children of σ, otherwise, this c will be lost.

- The rule $s\ a \rightarrow_{max} s'\ b\ (c)_{\downarrow_{\sigma'}}$ (where $\sigma' \in K$) can be applied and, after being applied twice, cell σ will contain bb and a copy of cc will be sent to σ', if σ' appears among the children of σ, otherwise, this cc will be lost.

Provided that all rules exclusively use the *min*, *max*, *repl*, and K operators, the system is guaranteed to be *deterministic*. However, operators such as *par*, *one*, *spread* may introduce *non-determinism*. In this paper, we are only interested in *deterministic solutions*, and we will exclusively use the *min*, *max*, *repl*, K operators.

By default, unless specifically mentioned, the channels are *duplex*, allowing simultaneous transmissions from both ends. *Simplex* channels are explicitly specified, and indicate a single open direction, either from parent to child, or from child to parent; messages sent in the other direction are "lost".

This definition of P module subsumes several earlier definitions of P systems, hP systems and nP systems. If δ is a *tree*, then P is essentially a tree-based P system (also known as cell-like P system). If δ is a *dag*, then P is essentially an hP system. If δ is a *digraph*, then P is essentially an nP system.

Given an arbitrary finite set of P modules, we can construct a higher level P module by creating channels between ports of the given P modules. This construction requires that the original P modules have disjoint cells.

Consider a finite family of n P modules, $\mathcal{P} = \{\Pi_i \mid i \in [1,n]\}$, where $\Pi_i = (O_i, K_i, \delta_i, P_i)$, $i \in [1,n]$. This family $\mathcal{P}$ is *cell-disjoint*, if their cell sets are disjoint, i.e., $K_i \cap K_j = \emptyset$, for $i, j \in [1,n]$. If required, any such family can be made cell-disjoint, by a *deep copy* process, which clones all cells and, as a convenience, automatically allocates successive indices to cloned cells (e.g., starting from cell σ, the first cloned cell is σ_1, the second is σ_2, etc).

Definition 2 (P module composition)**.** The P module $\Psi = (O, K, \delta, P)$ is a *composition* of the P module family $\mathcal{P}$, if:

- $\mathcal{P}$ is cell-disjoint,
- $O = \bigcup_{i\in[1,n]} O_i$,
- $K = \bigcup_{i\in[1,n]} K_i$,
- $\delta = \delta' \cup \bigcup_{i\in[1,n]} \delta_i$, where δ' is a binary relation on $\bigcup_{i\in[1,n]} P_i$,
- $P \subseteq \bigcup_{i\in[1,n]} P_i$.

In this case, the P modules in $\mathcal{P}$ are called *components* of Ψ.

When defining a new P module composition, we only need two items: (1) the additional δ' relation and (2) the remaining port set P. To simplify the discourse, we will use this approach, and omit the description of the other components, which are always the same in any P module composition.

This modular approach provides encapsulation, information hiding and recursive composition, facilitating the design of P programs for complex algorithms.

The following definition embodies the idea of a P system with rules which depend on generic objects, which can be specified at a later stage.

Definition 3 (Generic P module)**.** A *generic P module* is a system $\Pi\langle x_1, x_2, \ldots, x_n\rangle = (O, K, \delta, P)$, where its generic parameters, $x_1, x_2, \ldots, x_n$, designate fixed, but yet unspecified, objects. These generic parameters can be used as additional objects in the definition of its rules and must be instantiated to actual objects in O, before the rules can be applied. For each cell, its rule sequence is also generic on $\langle x_1, x_2, \ldots, x_n\rangle$, emphasizing that these additional symbols can be used as objects.

Instantiation is indicated by assigning specific objects to generic parameter names, and is accomplished by an automatic *deep copy* plus a *textual substitution* of the generic parameter names by their associated specific objects. We accept both *total instantiations*, which fix all generic parameters of a generic P module, and *partial instantiations*, which only fix a subset of the generic parameters.

For example, consider a generic P module $\Pi\langle x, y\rangle = (O, \{\sigma_{xy}, \tau_{xy}\}, \{\sigma_{xy} \rightarrow \tau_{xy}\}, \{\sigma_{xy}\})$, where x and y are its generic parameters, and assume that the object set O includes the digits. Then, $\Pi\langle x = 2, y = 3\rangle = (O, \{\sigma_{23}, \tau_{23}\}, \{\sigma_{23} \rightarrow \tau_{23}\}, \{\sigma_{23}\})$ is a total instantiation, and $\Pi\langle x, y = 3\rangle = (O, \{\sigma_{x3}, \tau_{x3}\}, \{\sigma_{x3} \rightarrow \tau_{x3}\}, \{\sigma_{x3}\})$ is a partial instantiation.

As suggested above, a good practice is to systematically index all cells of a generic P module, by the names of the generic parameters or of their actual associated objects (or their indices). Although not required, we will generally follow this convention.

Composing generic P modules, or a mixture of generic and non-generic P modules, constructs another generic P module. Depending on the intended effect, generic parameter names can be freely renamed (or not), as needed. For example, we can combine $\Pi\langle x = 3, y\rangle$, $\Pi\langle x = 4, y\rangle$ into a generic P module $\Gamma\langle b\rangle$; and $\Pi\langle x = 3, y_1\rangle$, $\Pi\langle x = 4, y_2\rangle$ into a generic P module $\Delta\langle y_1, y_2\rangle$.

While generic P modules are not strictly needed, we will use them to better manage the design complexity.

3 Byzantine agreement problem

We first introduce an anthropomorphic version of the Byzantine agreement problem:

> The Byzantine Generals' Problem is an agreement problem (first proposed by Pease *et al.* [PSL80]) in which N generals of the Byzantine Empire's army must unanimously decide whether to attack some enemy army or to retreat.
>
> The problem is complicated by the geographic separation of the generals, who must communicate by sending messengers to each other, and by the possible presence of up to F traitors amongst the N generals. These traitors can act arbitrarily in order to achieve the following aims: trick some generals into attacking; force a decision that is not consistent with the generals' desires, e.g., forcing an attack when no general wished to attack; or confusing some generals to the point that they are unable to make up their minds. If the traitors succeed in any of these goals, any resulting attack is doomed, as only a concerted effort can result in victory.
>
> Byzantine fault tolerance can be achieved if the loyal (non-faulty) generals have a unanimous agreement on their strategy. Note that if all loyal generals start with the same initial assessment value, attack or retreat, they must in the end agree upon the same value. Otherwise, the choice of strategy agreed upon is irrelevant.
>
> The Byzantine failure assumption models real-world environments in which computers and networks may behave in unexpected ways due to hardware

> failures, network congestion and disconnection, as well as malicious attacks. Byzantine failure-tolerant algorithms must cope with such failures and still satisfy the specifications of the problems they are designed to solve. Such algorithms are commonly characterized by their resilience F, the number of faulty processes with which an algorithm can cope.
>
> [Taken from [Wik09a], with minor changes]

In less anthropomorphic terms, we are given N distributed participants (or processes), which model the N generals. A small number F of these participants model the traitors and are called Byzantine faulty, i.e., they can fail in any possible way. All the faults considered here are Byzantine faults and we will use the abbreviations *fault* and *faulty* to mean Byzantine fault and Byzantine faulty, respectively. Example of faults are: sending different messages to different participants, sending incorrect messages, refraining from sending messages. Briefly, it can do anything that has a chance of disrupting the agreement. The other $N - F$ participants model the loyal generals and are called correct. Correct participants never fail and strictly follow the same agreement algorithm. The initial decision values are conventionally represented by a single bit, e.g., 1 for "attack" and 0 for "retreat".

In this paper, we consider only the basic scenario, where each pair of participants is connected by a fully reliable duplex channel and the resulting network works synchronously. It is well known that, in this basic case, the agreement is possible, if and only if $N \geq 3F + 1$. Note that, outside this basic scenario, the agreement is still possible for $2F + 1$ connected communication graphs, for channels with specific bounds on faults, for asynchronous networks with specific bounds on delays. However, agreement is not possible for arbitrary communication graphs, for arbitrary communication faults or for unbounded delays. These topics are not further discussed here and, for further details, refer to Lynch [Lyn96].

Figure 1 illustrates this basic case. We have four participants, P_1, P_2, P_3, P_4, with initial values 0, 0, 1, 1, respectively. These four participants are connected in a complete graph, often with loopback arcs (useful for uniform treatment), where the arcs indicate fully reliable channels. We assume that P_2, P_3, P_4 are correct, but P_1 is faulty. In this case, correct participants can agree, because $N = 4$, $F = 1$, and $N \geq 3F + 1$. Each participant has its own exact copy of one of the existing algorithms which solves the Byzantine agreement problem. In the next section, we review the first classical algorithm for this problem and illustrate how the agreement is always reached, despite the effort of the faulty participant P_1.

4 Classical Byzantine agreement algorithm based on EIG trees

The classical solution of the Byzantine agreement problem uses EIG $T_{N,L}$ trees as a data structure, and guarantees a solution if $N \geq 3F + 1$ and $L = F + 1$. It also uses a built-in default value, W (called V_0 [Lyn96]), to break ties and to replace wrong or missing messages. These parameters N, L and W are global and "hardcoded" into its rules.

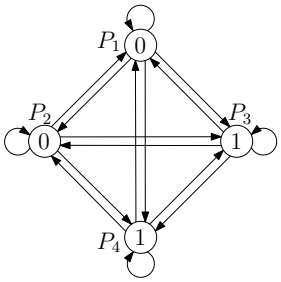

Figure 1: A Byzantine agreement problem, with $N = 4$.

A complete description of this algorithm is available in Lynch [Lyn96]. We give a simplified description, illustrating this on a particular case, where $N = 4$, $F = 1$, $L = 2$, $W = 0$. We assume that participants P_1, P_2, P_3, P_4 start with the initial values 0, 0, 1, 1, respectively, as shown in Figure 1. Participants P_2, P_3, P_4 are assumed correct, but P_1 is faulty and therefore allowed to send out arbitrary messages, if it chooses so.

This algorithm works in two distinct phases: first, a messaging phase, where the EIG trees are populated with the received messages, in a top-down order and, secondly, an evaluation phase which works bottom-up on the EIG trees.

We use apostrophes ($'$) and quotation marks ($''$) to mark top-down values and bottom-up values, respectively (Lynch designates these values *val* and *newval* [Lyn96]).

4.1 Phase I: messaging and filling top-down values

Phase I consists of L messaging rounds, which fill the EIG trees, top-down, one tree level per round. In our specific example, there will be two messaging rounds, respectively filling the first and the second EIG tree levels.

In the first messaging round, each correct participant, P_i, sends a copy of its initial decision value, v_i, to each of all four participants, i.e., to the other three participants and, using a loopback interface, back to itself, $P_i \overset{v_i}{\Rightarrow} P_j$, $j \in [1, 4]$. For example, the correct participant P_2, with initial value $v_0 = 0$, sends out four identical 0 messages to all four participants: $P_2 \overset{0}{\Rightarrow} P_i$, $i \in [1, 4]$. The other two correct participants, P_3 and P_4, proceed similarly: $P_3 \overset{1}{\Rightarrow} P_i$, $i \in [1, 4]$, $P_4 \overset{1}{\Rightarrow} P_i$, $i \in [1, 4]$.

A correct participant P_1 would have also been expected to send out identical messages to all participants, according to its initial decision value, 0 in our example. However, P_1 is faulty and can send out conflicting messages, if it wishes so. For example, P_1 sends 0 to each of P_1, P_2, P_3, but 1 to P_4: $P_1 \overset{0}{\Rightarrow} P_i$, $i \in [1, 3]$, $P_1 \overset{1}{\Rightarrow} P_4$.

As the channels are all reliable, all messages are properly received. Participant P_2 receives the following four messages: $P_2 \overset{0}{\Leftarrow} P_1$, $P_2 \overset{0}{\Leftarrow} P_2$, $P_2 \overset{1}{\Leftarrow} P_3$, $P_2 \overset{1}{\Leftarrow} P_4$. Participants P_1 and P_3 receive the same messages as P_2, for example: $P_3 \overset{0}{\Leftarrow} P_1$, $P_3 \overset{0}{\Leftarrow} P_2$, $P_3 \overset{1}{\Leftarrow} P_3$, $P_3 \overset{1}{\Leftarrow} P_4$. However, one the messages received by P_4 differs: $P_4 \overset{1}{\Leftarrow} P_1$, $P_4 \overset{0}{\Leftarrow} P_2$, $P_4 \overset{1}{\Leftarrow} P_3$, $P_4 \overset{1}{\Leftarrow} P_4$.

All round 1 received values are now stored in the EIG trees, in a position related to the sender's identity, which is known by the receiver. Each correct participant P_k uses

its EIG node i to store the value v received from participant P_i. For example, P_2 stores $0'$, $0'$, $1'$ and $1'$, in its EIG nodes 1, 2, 3 and 4, respectively, as also shown by level 1 EIG nodes of Figure 2. Level 1 EIG nodes of Figures 3 and 4 illustrate the round 1 messages received and stored by P_3 and P_4, respectively.

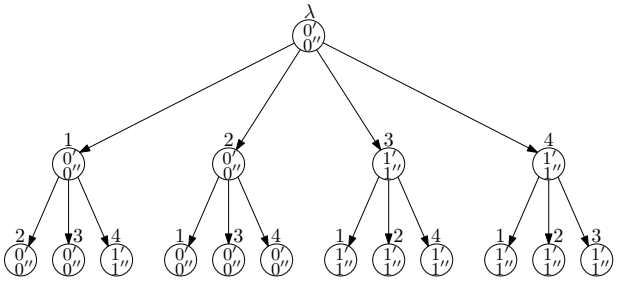

Figure 2: EIG tree for P_2.

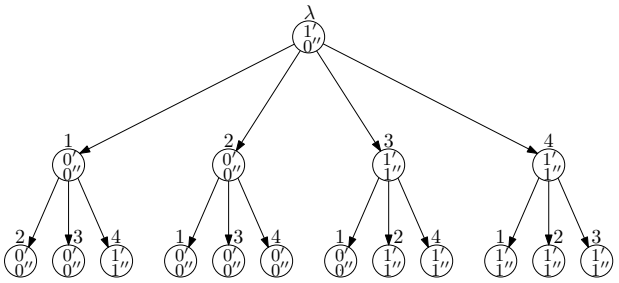

Figure 3: EIG tree for P_3.

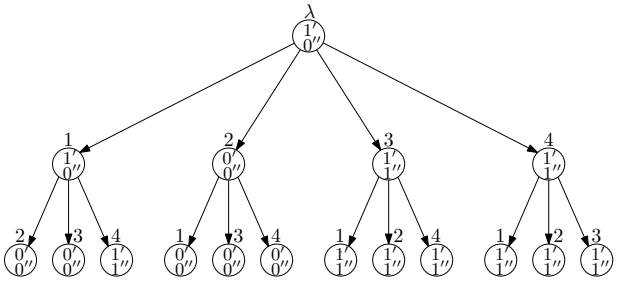

Figure 4: EIG tree for P_4.

In the second messaging round, each correct participant further relays copies of round 1 received messages, to all four participants. All received messages are faithfully relayed, except where this would create loops. For this round, this means that a participant will not relay messages that have been received via its own loopback interface (i.e., a message originated from itself at round 1). For example, participant P_2 will not further relay the message 0 received from itself: $P_2 \overset{0}{\Rightarrow} P_2$.

The messages are sent using a protocol that identifies the original source of each message. Although not size optimal, we will here assume a straightforward protocol,

which prefixes each message with the ID of the originator. For example, P_2 sends out four identical messages to all four participants: $P_2 \overset{(1,0)(3,1)(4,1)}{\Longrightarrow} P_i$, $i \in [1, N]$. The other two correct participants, P_3 and P_4, proceed in a similar way: $P_3 \overset{(1,0)(2,0)(4,1)}{\Longrightarrow} P_i$, $i \in [1, 4]$, $P_4 \overset{(1,0)(2,0)(3,1)}{\Longrightarrow} P_i$, $i \in [1, 4]$.

Again, our faulty participant P_1 can, if it wishes, send out conflicting messages, which may or may not be consistent with its received values. For example, consider that P_1 sends out the following round 2 messages: $P_1 \overset{(2,0)(3,0)(4,1)}{\Longrightarrow} P_3$, $P_1 \overset{(2,0)(3,1)(4,1)}{\Longrightarrow} P_i$, $i \in \{1, 2, 4\}$.

All messages are properly received and stored in level 2 EIG nodes. Each correct participant P_k uses its EIG node $i.j$ to store the value v received from P_j via the message $P_j \overset{(i,v)}{\Longrightarrow} P_k$. For example, participant P_2 stores the values $0'$, $0'$, $0'$, $0'$, $0'$, $0'$, $1'$, $1'$, $1'$, $1'$, $1'$, $1'$, in its EIG nodes 1.2, 1.3, 1.4, 2.1, 2.3, 2.4, 3.1, 3.2, 3.4, 4.1, 4.2, 4.3, respectively. Level 2 EIG nodes of Figures 2, 3 and 4 illustrate the round 2 messages received and stored by participants P_2, P_3 and P_4.

In our case, the messaging rounds end after filling two levels in the EIG trees. However, in general, messaging will continue, using a similar mechanism, until all EIG levels are completely filled. Essentially, each correct participant P_k will use its EIG node $i_1.i_2 \ldots i_t.i_{t+1}$ to store the value v received from $P_{i_{t+1}}$ via the message $P_{i_{t+1}} \overset{(i_1.i_2 \ldots i_t, v)}{\Longrightarrow} P_k$. A correct participant $P_{i_{t+1}}$ will forward such a message only if it does not create a loop, i.e., if $i_{t+1} \notin \{i_1, i_2, \ldots, i_t\}$. The recursive application of this loop avoiding strategy ensures that the sequence $i_1.i_2 \ldots i_t.i_{t+1}$ is one of the permutations of $[1, N]$ of size $t + 1$ and the value v always finds its proper unique place in the EIG tree.

Intuitively, this v is claimed to be the initial value of P_{i_1}, further relayed to P_k via P_{i_2}, ..., P_{i_t}, $P_{i_{t+1}}$, in this order. In fact, this is indeed the case, if all these participants are correct. For further and more precise details, see Lynch [Lyn96].

4.2 Phase II: evaluating bottom-up values

Phase II consists of L evaluation rounds, which proceed level by level, in a bottom-up manner.

First, for a leaf EIG node, the bottom-up value is set equal to its already filled top-down value. In our example, for participant P_2, the EIG nodes 1.2, 1.3, 1.4, 2.1, 2.3, 2.4, 3.1, 3.2, 3.4, 4.1, 4.2, 4.3, evaluate the following bottom-up values: $0''$, $0''$, $0''$, $0''$, $0''$, $0''$, $1''$, $1''$, $1''$, $1''$, $1''$, $1''$, respectively.

Next, assume that the bottom-up values have already been evaluated for level $L - t$, $t \in [0, L - 1]$. The bottom-up values at the next higher level, $L - t - 1$, are evaluated using a strict majority rule, or, if there is no strict majority, the result is the default value W (0 in our case). For example, using the strict majority rule, participant P_2's EIG nodes 1, 2, 3, 4, evaluate the bottom-up values $0''$, $0''$, $1''$, $1''$, respectively. However, at the next round, no strict majority exists at the EIG node λ. This tie is broken using the default value $0''$. Because λ is the EIG root, the final value for P_2 is 0.

Figures 2, 3 and 4 illustrate all bottom-up values evaluated by participants P_2, P_3 and P_4, respectively. One can see that all correct participants reach a common final decision. Although this is not required by the formal specifications of the Byzantine

agreement problem, this common decision can also be reached by the faulty participant P_1, regardless of its arbitrary outgoing messages, if it bothers to properly fill and evaluate an EIG tree. For further and more precise details, again see Lynch [Lyn96].

This brief example illustrates some fundamental properties of the EIG Byzantine agreement algorithm. The correct participants always reach a common decision, as long as the number of faulty participants does not exceed the prescribed bound F (here $F = 1$). In some border cases, by "cleverly" sending out inconsistent messages, the faulty participants are able to sway the common decision one way or another, but never to disrupt it.

We can show this by reconsidering the above example, with the only difference that, at the first messaging round, the faulty participant P_1 sends out 1 (instead of 0) to P_3, $P_1 \stackrel{1}{\Rightarrow} P_3$. In this case, all correct participants, P_2, P_3, P_4, will all reach the final decision 1 (instead of 0). They will still agree on a common decision value.

5 P system program for the Byzantine agreement

The following global parameters are known in advance and "hard-coded" into our current model: N, the number of participants, L, the height of the EIG trees, and W, the default value, for wrong or missing values.

We design our program by recursive composition of simpler P modules. The common vocabulary, O, used by all P modules includes the set $\{v, v', v'' \mid v \in \{0,1\}\} \cup \{?, *\} \cup \{x^v_\pi \mid v \in \{0,1,?\}, \pi \in P(N,t), t \in [0,L]\}$. Depending on the objects sent by faulty P modules, O can be larger than this set, as we cannot constrain the behaviour of faulty participants in any way.

Objects 0 and 1 designate decision values. Objects $0'$ and $1'$ represent decision values stored as top-down values in the EIG trees. Objects $0''$ and $1''$ represent decision values stored as bottom-up values in the EIG trees. Object ? is a template that can match any decision value, 0 or 1. Object $*$ designates the last step in the top-down evaluation.

The object $x^v_{i_1.i_2...i_t}$ represents a t^{th} round message received from P_{i_t}, i.e., using our earlier notation, $x^v_{i_1.i_2...i_t} = (i_1.i_2 \ldots i_t, v)$, where the right-hand side string is considered a single object. To simplify the notations, we also use the following natural conventions: $x^v_{i_1.i_2...i_t} = x^v_{i_1 i_2...i_t}$ and $x^v_{()} = x^v = v$. These notations are summarized in Figure 5. We prefer the x^v_π notation when we want to emphasize an "atomic" vocabulary object, and the (π, v) notation when we work on its constituent "sub-atomic" objects.

$$\begin{array}{lcl} x^v & = & v, \text{ for } v \in \{0,1\} \\ x^v_\pi & = & (\pi, v), \text{ for } v \in \{0,1\}, \\ & & t \in [0,L], \ \pi \in P(N,t) \end{array}$$

Figure 5: Notations summary (left, "atomic" notation; right, "sub-atomic" insight).

Our design uses the following elementary P modules: Ψ, a P module representing

the "core" of a participant in a Byzantine decision; Θ, a P module representing an EIG tree; and $\Gamma\langle h, f\rangle$, a generic P module, which takes care of all communications of participant h, with another participant f, if $h \neq f$, or with self, otherwise. Using these elementary P modules, as basic building blocks, we compose the generic P module $\Pi\langle h\rangle$, which represents a correct participant with index h, and, finally, the P module Ω, which represents a complete Byzantine scenario.

Figure 6 illustrates the generic P module $\Pi\langle h\rangle$, for the case $N = 4$ and $L = 2$, including its constituent P modules: Ψ, Θ, $\Gamma\langle h, f = 1\rangle$, $\Gamma\langle h, f = 2\rangle$, $\Gamma\langle h, f = 3\rangle$, $\Gamma\langle h, f = 4\rangle$. Dotted lines represent P module borders and shaded cells are the remaining ports of the figure's top P module, $\Pi\langle h\rangle$. As this figure clearly shows, Θ is (as expected) based on a *tree*, which is further included in the *dag* underlying the participant $\Pi\langle h\rangle$. The rest of this section clarifies this construction. We will first focus on the structural details and consider the rules after these are completed.

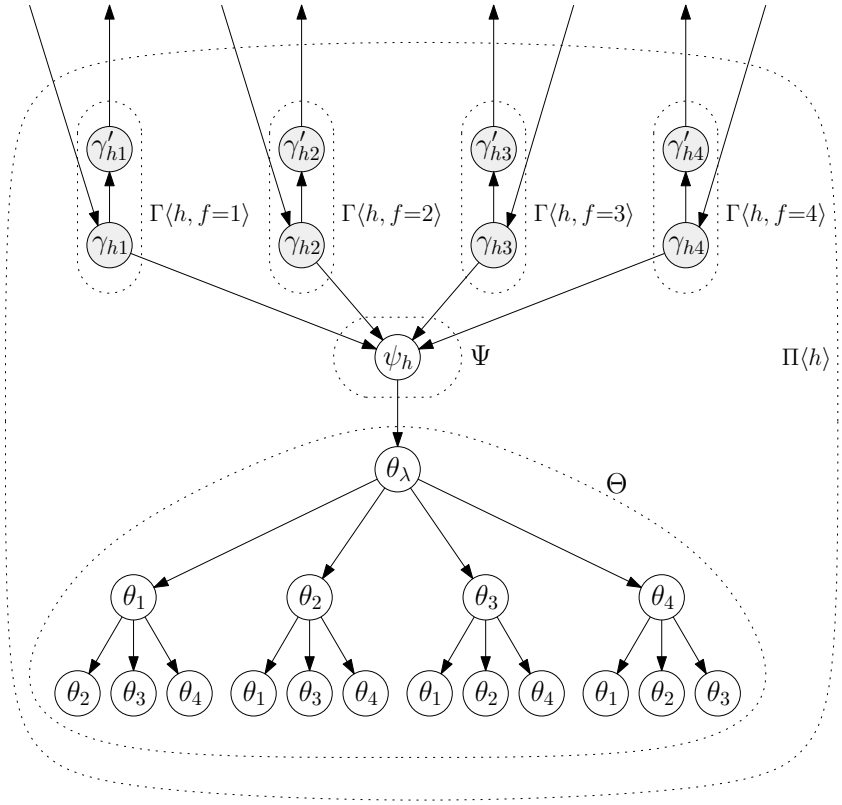

Figure 6: The P module $\Pi\langle h\rangle$, for $N = 4$, $L = 2$.

The P module Ψ contains a single cell and is defined by: $\Psi = (O, K_\Psi, \emptyset, P_\Psi)$, where $K_\Psi = P_\Psi = \{\psi\}$, $\psi = (Q_\psi, s_0, v, R_\psi)$, $Q_\psi = \{s_i \mid i \in [0, L]\} \cup \{s_z\}$, $v \in \{0, 1\}$ is the initial decision value of this participant and the rule sequence R_Ψ is given in the next section. The P module Ψ does not need to be generic, its rules are identical for all participants.

The P module Θ contains the EIG tree and is defined by: $\Theta = (O, K_\Theta, \delta_\Theta, P_\Theta)$. Essentially, K_Θ and δ_Θ define an EIG tree as previously described, for the global parameters N and L. The root cell of the tree is labelled θ_λ, which is also its single

port, $P_\Theta = \{\theta_\lambda\}$. All EIG cells start with empty contents and share the same states, $Q_\Theta = \{d_t \mid t \in [0, L]\} \cup \{u_t \mid t \in [0, 5]\} \cup \{t_z\}$, and rule sequence R_Θ, which is given in the next section. The P module Θ does not need to be generic, its rules are identical for all participants.

The generic P module $\Gamma\langle h, f\rangle$ contains two cells and is defined by: $\Gamma\langle h, f\rangle = (O, K_\Gamma, \delta_\Gamma, P_\Gamma)$, where $K_\Gamma = P_\Gamma = \{\gamma_{hf}, \gamma'_{hf}\}$, $\delta_\Gamma = \{\gamma_{hf} \to \gamma'_{hf}\}$, $\gamma_{hf} = (Q_\gamma, p_0, \emptyset, R_\gamma\langle h, f\rangle)$, $Q_\gamma = \{p_t, q_t, r_t \mid t \in [0, L-1]\} \cup \{p_L, p_z\}$, $\gamma'_{hf} = (Q'_\gamma, c_0, \emptyset, R'_\gamma)$, $Q'_\gamma = \{c_t \mid t \in [0, 3]\}$.

The rule sequences $R_\gamma\langle h, f\rangle$ (generic) and R'_γ (non-generic) are given in the next section. After constructing the higher levels P modules (Π and Ω), these generic parameters will be fixed: $h \in [1, N]$, as the index of the participant which contains it (its "home"); and $f \in [1, N]$, as the index of the participant at the other connection end (a potentially faulty "friend-or-foe"). Although not strictly necessary, this generic approach facilitates a uniform design.

We now design a higher generic P module, $\Pi\langle h\rangle = (O, K_\Pi, \delta_\Pi, P_\Pi)$, representing a generic Byzantine participant, with index h, by composing: one deep copy of P module Ψ (the main cell), one deep copy of P module Θ (the EIG tree), and the following N partial instances of the generic P module $\Gamma\langle h, f\rangle$: $\Gamma\langle h, f = 1\rangle$, $\Gamma\langle h, f = 2\rangle$, $\ldots$, $\Gamma\langle h, f = N\rangle$. To complete the definition, we define its additional arcs, $\delta'_\Pi = \{\psi_h \to \theta_\lambda\} \cup \{\gamma_{hj} \to \psi_h \mid j \in [1, N]\}$, and its remaining port set, $P_\Pi = \{\gamma_{hj} \mid j \in [1, N]\}$.

It might be useful, at this stage, to have a second look at Figure 6. In this case, $N = 4$ and the P module $\Pi\langle h\rangle$ has four groups of two ports available for further connections—one group for each participant (including self). Single ended arrows indicate how this participant will be finally connected.

To complete the design, we define the final composition, $\Omega = (O, K_\Omega, \delta_\Omega, P_\Omega)$, representing a complete Byzantine scenario, by composing the following N instances of the P module $\Pi\langle h\rangle$: $\Pi\langle h = 1\rangle$, $\Pi\langle h = 2\rangle$, $\ldots$, $\Pi\langle h = N\rangle$. We define its remaining port set, $P_\Omega = \emptyset$ (assuming that Ω does not need to be further connected); and its additional arcs, $\delta'_\Omega = \{\gamma'_{ij} \to \gamma_{ji} \mid i, j \in [1, N]\}$, where these new arcs are *simplex* channels, only from transmission from child (γ_{ij}) to parent (γ'_{ij}). At this stage, we have completed the customization of parameters f and h, for each constituent $\Gamma\langle i, j\rangle$, $i, j \in [1, N]$.

Figure 7 illustrates a fragment of the P module Ω (for $N = 4$ and $L = 2$) showing the channels between two participants, Π_2 and Π_3; all other connection pairs are similar. As expected, the last added channels define a complete graph among the participants, where each participant has also a *loopback* connection, corresponding to the channels $\gamma'_{ii} \to \gamma_{ii}$, $i \in [1, N]$.

To start the system, we "magically" drop the initial decision values in participants' main cells, ψ_i, for $i \in [1, N]$. Thereafter, all cells start in the same time step, synchronously, as in the standard Byzantine agreement algorithms [Lyn96].

Remark 4. The number of cells in the P module Π grows exponentially with N, the number of participants, and L, the height of the EIG trees used. It is easy to see by induction, that each EIG tree $T_{N,L}$ contains at most $2(N)_L$ nodes, where $(N)_L$ denotes the falling factorial $N(N-1)\cdots(N-L-1)$. Thus, since our P module construction needs N copies, we have an upper bound of $O((N)_L N)$ number of cells.

As mentioned earlier, in any Byzantine agreement algorithm, the maximum number

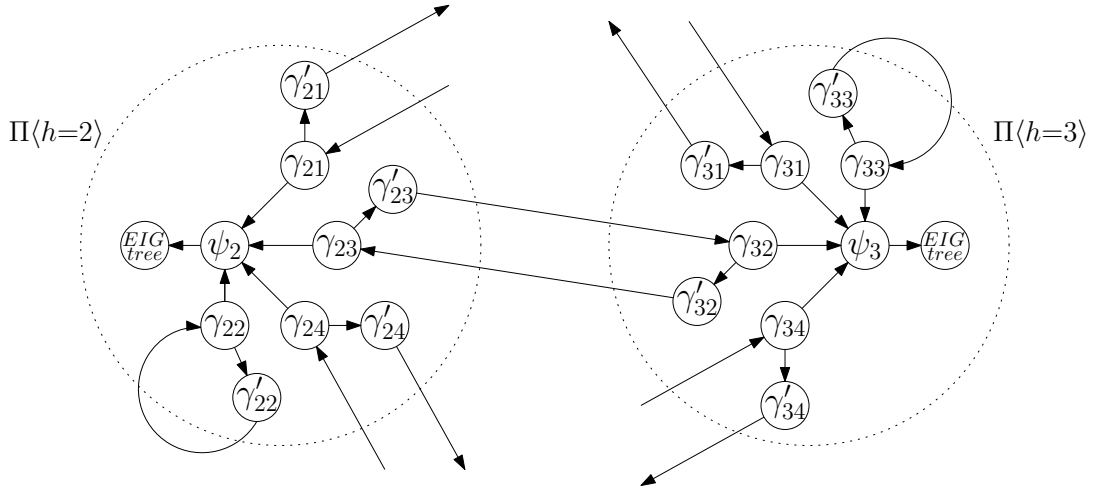

Figure 7: Fragment of the P module Ω, for $N = 4$, $L = 2$, showing connections between participants 2 and 3 (here, node contents indicate cell indices).

of tolerated faults is $F = (N-1)/3$. Also, in the EIG algorithm, the maximum number of tolerated faults is bounded by the height of the EIG tree, $F \leq L-1$. Therefore, for maximum fault tolerance, L is linearly related to N, $L = (N-1)/3+1$, and therefore we conclude that the number of cells in Π grows exponentially with N.

6 Byzantine Agreement Rules

We summarize the rule sequences as templates, which (most of them) depend on the "hard-coded" global parameters N, L and W. As earlier mentioned, several rules for $R_\gamma\langle h, f\rangle$ depend additionally on the generic objects h ("home") and f ("friend-or-foe"). Note that, for some rule templates, the number of corresponding actual rules grows exponentially with N and L.

To facilitate the understanding of our rules, each rule sequence is preceded by a statechart, graphically illustrating state transitions. Where several rules are grouped together, their relative order is omitted as irrelevant, because they start from the the same state, end in the same state, and their left-hand sides are disjoint. Also, each rule template, which refers to permutations of size L or $L-1$, is followed by an itemized expansion for the sample case $L = 2$ (as used by our specific examples).

1. Rule sequences for $R_\gamma\langle h, f\rangle$ and R'_γ (i.e., for the port cells of $\Pi\langle h\rangle$):

   1.1 The state transitions for these sequences are given in Figure 8.

   1.2 Note that state p_z is a final state for cell γ_{hf}, i.e., no further transition is possible from this state. However, cell γ'_{hf} does not have such a final state; cell γ'_{hf} will normally end in state c_2.

   1.3 $c_0 \rightarrow_{min} c_1$

   1.4 $c_0\ o \rightarrow_{max} c_1$, for $o \in O$

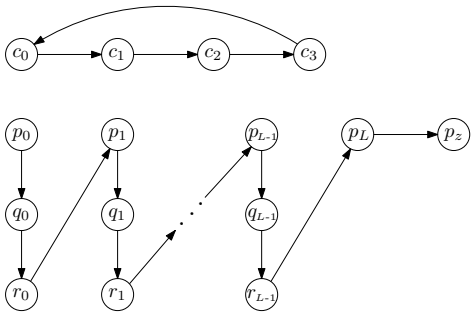

Figure 8: State diagram for $\Gamma\langle h, f\rangle$.

1.5 $c_1 \rightarrow_{min} c_2$

1.6 $c_2\ x_\pi^v \rightarrow_{min} c_3\ (x_\pi^v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$ and $\pi \in \bigcup_{l\in[0,L-1]} P(N,l)$

- $c_2\ v \rightarrow_{min} c_3\ (v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$
- $c_2\ x_j^v \rightarrow_{min} c_3\ (x_j^v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, $j \in [1,N]$

1.7 $c_3 \rightarrow_{min} c_0$

1.8 $p_t\ x_\pi^v \rightarrow_{min} q_t$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h = f$, $h \in Im(\pi)$

- $p_1\ x_j^v \rightarrow_{min} q_1$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h = f$, $h = j$

1.9 $p_t\ x_\pi^v \rightarrow_{min} q_t\ x_\pi^?\ (x_\pi^v)_{\uparrow_{repl}}$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h = f$, $h \notin Im(\pi)$

- $p_0\ v \rightarrow_{min} q_0\ ?\ (v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, $h = f$
- $p_1\ x_j^v \rightarrow_{min} q_1\ x_j^?\ (x_j^v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h = f$, $h \neq j$

1.10 $p_t\ x_\pi^v \rightarrow_{min} q_t$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h \neq f$, $h \in Im(\pi)$, $f \in Im(\pi)$

- $p_1\ x_j^v \rightarrow_{min} q_1$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h \neq f$, $h = j$, $f = j$

1.11 $p_t\ x_\pi^v \rightarrow_{min} q_t\ x_\pi^?$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h \neq f$, $h \in Im(\pi)$, $f \notin Im(\pi)$

- $p_1\ x_j^v \rightarrow_{min} q_1\ x_j^?$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h \neq f$, $h = j$, $f \neq j$

1.12 $p_t\ x_\pi^v \rightarrow_{min} q_t\ (x_\pi^v)_{\uparrow_{repl}}$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h \neq f$, $h \notin Im(\pi)$, $f \in Im(\pi)$

- $p_1\ x_j^v \rightarrow_{min} q_1\ (x_j^v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h \neq f$, $h \neq j$, $f = j$

1.13 $p_t\ x_\pi^v \rightarrow_{min} q_t\ x_\pi^?\ (x_\pi^v)_{\uparrow_{repl}}$, for $t \in [0, L-1]$, $v \in \{0,1\}$, and $\pi \in P(N,t)$, s.t. $h \neq f$, $h \notin Im(\pi)$, $f \notin Im(\pi)$

- $p_0\ v \rightarrow_{min} q_0\ ?\ (v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, $h \neq f$
- $p_1\ x_j^v \rightarrow_{min} q_1\ x_j^?\ (x_j^v)_{\uparrow_{repl}}$, for $v \in \{0,1\}$, and $j \in [1,N]$, s.t. $h \neq f$, $h \neq j$, $f \neq j$

1.14 $p_L \rightarrow_{min} p_z$

1.15 $q_t \rightarrow_{min} r_t$, for $t \in [0, L-1]$

1.16 $r_t\ x^?_\pi\ x^0_\pi \rightarrow_{min} p_{t+1}\ (x^0_{\pi\oplus(f)})_{\downarrow_{repl}}$, for $t \in [0, L-1]$, and $\pi \in P(N,t)$, s.t. $f \notin Im(\pi)$

- $r_0\ ?\ 0 \rightarrow_{min} p_1\ (x^0_f)_{\downarrow_{repl}}$,
- $r_1\ x^?_j\ x^0_j \rightarrow_{min} p_2\ (x^0_{jf})_{\downarrow_{repl}}$, for $j \in [1, N]$, s.t. $f \neq j$

1.17 $r_t\ x^?_\pi\ x^1_\pi \rightarrow_{min} p_{t+1}\ (x^1_{\pi\oplus(f)})_{\downarrow_{repl}}$, for $t \in [0, L-1]$, and $\pi \in P(N,t)$, s.t. $f \notin Im(\pi)$

- $r_0\ ?\ 1 \rightarrow_{min} p_1\ (x^1_f)_{\downarrow_{repl}}$,
- $r_1\ x^?_j\ x^1_j \rightarrow_{min} p_2\ (x^1_{jf})_{\downarrow_{repl}}$, for $j \in [1, N]$, s.t. $f \neq j$

1.18 $r_t\ x^?_\pi \rightarrow_{min} p_{t+1}\ (x^W_{\pi\oplus(f)})_{\downarrow_{repl}}$, for $t \in [0, L-1]$ and $\pi \in P(N,t)$, s.t. $f \notin Im(\pi)$

- $r_0\ ? \rightarrow_{min} p_1\ (x^W_f)_{\downarrow_{repl}}$,
- $r_1\ x^?_j \rightarrow_{min} p_2\ (x^W_{jf})_{\downarrow_{repl}}$, for $j \in [1, N]$, s.t. $f \neq j$

1.19 $r_t\ y \rightarrow_{max} p_{t+1}$, for $t \in [0, L-1]$ and $y \in V$

2. Rule sequence R_Ψ (for the main cell of $\Pi\langle h\rangle$):

2.1 The state transitions for these sequences are given in Figure 9.

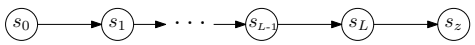

Figure 9: State diagram for the P module Ψ.

2.2 Note that state t_z is a final state for cell ψ_h, i.e., no further transition is possible from this state.

2.3 $s_t\ x^v_\pi \rightarrow_{min} s_{t+1}\ (x^v_\pi)_{\updownarrow_{repl}}$, for $t \in [0, L-1]$, $v \in \{0,1\}$ and $\pi \in P(N,t)$

- $s_0\ v \rightarrow_{min} s_1\ (v)_{\updownarrow_{repl}}$, for $v \in \{0,1\}$
- $s_1\ x^v_j \rightarrow_{min} s_2\ (x^v_j)_{\updownarrow_{repl}}$, for $v \in \{0,1\}$, $j \in [1, N]$

2.4 $s_L\ x^v_\pi \rightarrow_{min} s_z\ (*\ x^v_\pi)_{\downarrow_{repl}}$, for $v \in \{0,1\}$ and $\pi \in P(N,L)$

- $s_2\ x^v_{jk} \rightarrow_{min} s_z\ (*\ x^v_{jk})_{\downarrow_{repl}}$, for $v \in \{0,1\}$ and $j, k \in [1, N]$, $j \neq k$

3. Rule sequence R_Θ (for the EIG cells of $\Pi\langle h\rangle$):

3.1 The state transitions for these sequences are given in Figure 10.

3.2 Note that state t_z is a final state for all EIG cells, i.e., no further transition is possible from this state.

3.3 $d_0\ *\ v \rightarrow_{min} u_0\ v'\ v$, for $v \in \{0,1\}$

3.4 $d_0\ v \rightarrow_{min} d_1\ v'$, for $v \in \{0,1\}$

3.5 $d_t\ *\ x^v_\pi \rightarrow_{min} u_0\ (*\ x^v_{\pi(2:t)})_{\downarrow_{\pi(1)}}$, for $t \in [1, L]$, $v \in \{0,1\}$ and $\pi \in P(N,t)$

- $d_1\ *\ x^v_j \rightarrow_{min} u_0\ (*\ v)_{\downarrow_j}$, for $v \in \{0,1\}$, $j \in [1, N]$

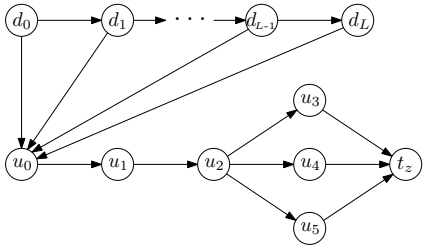

Figure 10: State diagram for the P module Θ.

- $d_2 \ * \ x^v_{jk} \rightarrow_{min} u_0 \ (* \ x^v_k)_{\downarrow_j}$, for $v \in \{0,1\}$, $j,k \in [1,N], j \neq k$

3.6 $d_t \ x^v_\pi \rightarrow_{min} d_{t+1} \ (x^v_{\pi(2:t)})_{\downarrow_{\pi(1)}}$, for $t \in [1, L-1]$, $v \in \{0,1\}$ and $\pi \in P(N,t)$

- $d_1 \ x^v_j \rightarrow_{min} d_2 \ (v)_{\downarrow_j}$, for $v \in \{0,1\}$, $j \in [1,N]$

3.7 $u_0 \ v \rightarrow_{min} u_1 \ v$, for $v \in \{0,1\}$

3.8 $u_1 \ 0 \ 1 \rightarrow_{max} u_2$

3.9 $u_1 \rightarrow_{min} u_2$

3.10 $u_2 \ 0 \rightarrow_{max} u_3$

3.11 $u_2 \ 1 \rightarrow_{max} u_4$

3.12 $u_2 \ \rightarrow_{min} u_5$

3.13 $u_3 \ \rightarrow_{min} t_z \ 0'' \ (0)_{\uparrow_{repl}}$

3.14 $u_4 \ \rightarrow_{min} t_z \ 1'' \ (1)_{\uparrow_{repl}}$

3.15 $u_5 \ \rightarrow_{min} t_z \ W'' \ (W)_{\uparrow_{repl}}$

7 Program correctness

Given the global parameters N, L and W, consider the following set definitions:

$$\begin{aligned}
P_g(N,t) &= \{\pi \in P(N,t) \mid g \notin Im(\pi)\}, \\
\aleph(t, v_1, v_2, \ldots, v_k) &= \{G(\omega) \mid \omega : P(N,t) \rightarrow \{v_1, v_2, \ldots, v_k\}\} \\
\aleph_g(t, v_1, v_2, \ldots, v_k) &= \{G(\omega) \mid \omega : P_g(N,t) \rightarrow \{v_1, v_2, \ldots, v_k\}\} \\
\beth(t, v_1, v_2, \ldots, v_k) &= \{x^v_\pi \mid v \in \{v_1, v_2, \ldots, v_k\}, \pi \in P(N,t)\}, \\
\beth_g(t, v_1, v_2, \ldots, v_k) &= \{x^v_\pi \mid v \in \{v_1, v_2, \ldots, v_k\}, \pi \in P_g(N,t)\}, \\
&\quad \text{where } g \in [1,N], \ t \in [0,L], \ v_1, v_2, \ldots, v_k \in V.
\end{aligned}$$

We first note a few straightforward remarks, linking these definitions to EIG trees, their top-down values, and the messages exchanged in the EIG-based Byzantine agreement algorithm, as described in Section 4. The set $\aleph(t, 0', 1')$ describes all possible assignments of $0'$ and $1'$ values to an EIG level t. The set $\aleph_g(t,0,1)$ describes all possible multisets which can be broadcast by a correct participant g (discarding permutations that will create circular links); the following relation holds: $\aleph_g(t,0,1) = \aleph(t,0,1) \cap \beth_g(t,0,1)$.

We begin with arguably the most complex and critical module $\Gamma\langle h, f\rangle$.

7.1 Runtime behaviour of P module $\Gamma\langle h, f\rangle$

Assume that this module is further connected to two unspecified external modules: h (representing a "home" side) and f (representing a "friend-or-foe" side). The input/ouput connection to h is realized by a duplex channel $\gamma_{hf} \rightarrow h$. The output connection to f is realized by a simplex channel $f \rightarrow \gamma_{hf}$ (one-way in the child-to-parent direction). The input connection from f is realized by a simplex channel $\gamma'_{hf} \rightarrow f$ (one-way in the child-to-parent direction).

Intuitively, this module takes care of all communication aspects between h and f. It expects that the messages from h are correct and "properly" forwards them to f. It expects arbitrary messages from f and "properly" filters and forwards them to h.

As its rules show, $\Gamma\langle h, f\rangle$ works in cycles of 4 steps, where each cycle corresponds to a messaging round of the Byzantine agreement algorithm described in Section 4. Figure 11 illustrates a typical cycle. Labels of small arrows indicate the steps when main messages are exchanged. Messages at steps 1, 2 and 4 are part of the external contract; the message at step 3 is an important but internal message. Broken small arrows indicate the one-way restrictions.

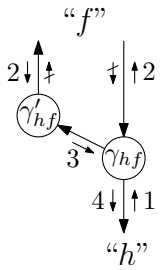

Figure 11: Contract for $\Gamma\langle h, f\rangle$.

Specifically, assuming the mentioned connections to h and f, this module obeys the following contract.

Contract 5 (External contract of $\Gamma\langle h, f\rangle$)**.** No other messages are externally sent or expected, except the ones explicitly mentioned below, where $t \in [0, L-1]$.

Inputs to γ_{hf} From h, port cell γ_{hf} expects a set $\omega^t_{hf} \in \aleph(t, 0, 1)$, in step $1 + 4t$.

Inputs to γ'_{hf} From f, port cell γ'_{hf} can receive any arbitrary external message, in any step. The arbitrary message, received in step $2 + 4t$, is designated as ϖ^t_{hf}.

Outputs from γ_{hf} To f, port cell γ_{hf} sends the message $\omega^t_{hf} \cap \beth_h(t, 0, 1) \in \aleph_h(y, 0, 1)$, in step $2+4t$. To h, port cell γ_{hf} sends the message $\mu^t_{hf}(\varpi^t_{hf})$, in step $4+4t$, where the filter μ^t_{hf} is described at the end of this contract.

The filter μ^t_{hf} is a mapping $\mu_{ft} : V^* \rightarrow \aleph_f(t, 0, 1)$, which can be described as follows. Given a (string representation of) multiset $z \in V^*$, $z' = \mu^t_{hf}(z)$ is constructed by examining each object $\pi \in P_f(N, t)$: if x^0_π occurs in z, with any multiplicity greater than 0, then one x^0_π is added to z', else, if x^1_π occurs in z, with any multiplicity greater than 0, then one x^1_π is added to z', else, one x^W_π is added to z' (where W is the default value).

A close examination of $\Gamma\langle h, f\rangle$'s rules shows that its contracts holds. The other modules can be assessed in a similar way. The details are not provided here.

Figures 12, 13 and 14 offer a complementary view of the overall behaviour of the Ω module, via partial traces of our main sample scenario, described in detail in the text of Section 4 and by Figures 2, 3 and 4.

Figure 12 illustrates traces, describing the messaging interaction between participants 2 and 3. The following cells are included: θ_{λ_2}, ψ_2, γ_{23}, γ'_{23} (for participant 2); and γ'_{32}, γ_{32}, ψ_3, θ_{λ_3} (for participant 3). After ten steps, all port cells reach their final states.

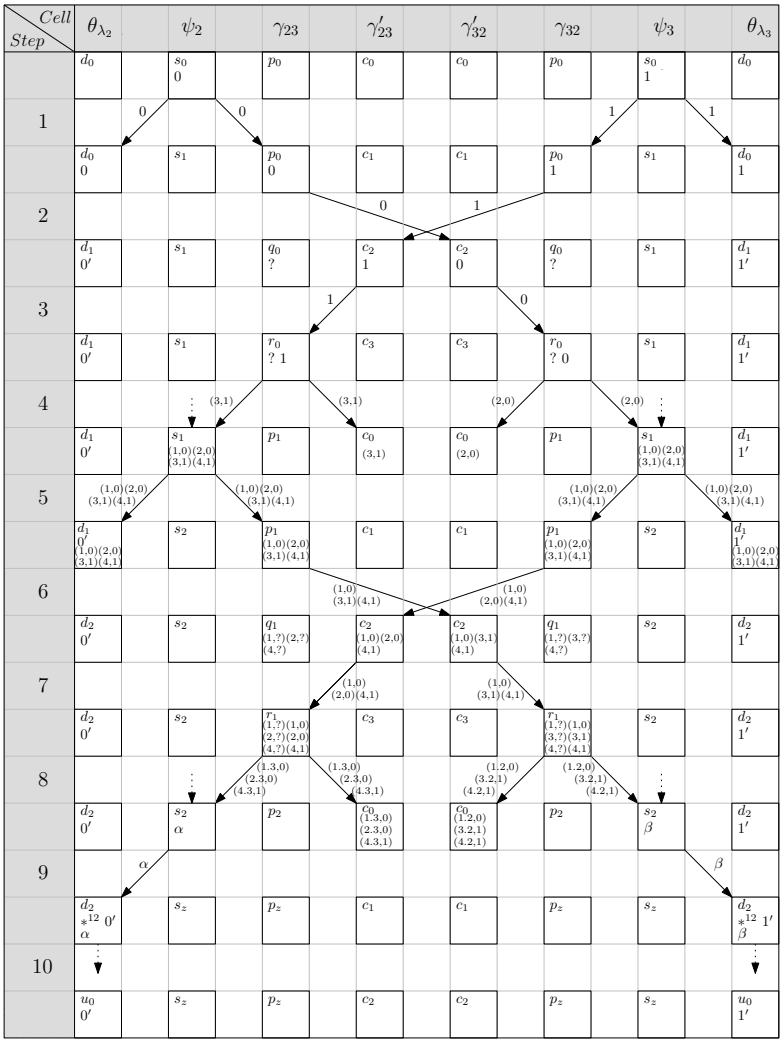

Figure 12: Traces of the message phase between participants 2 and 3 (fragments). Here, α = { (1.2,0), (1.3,0), (1.4,1), (2.1,0), (2.3,0), (2.4,0), (3.1,1), (3.2,1), (3.4,1), (4.1,1), (4.2,1), (4.3,1) } and β = { (1.2,0), (1.3,0), (1.4,1), (2.1,0), (2.3,0), (2.4,0), (3.1,0), (3.2,1), (3.4,1), (4.1,1), (4.2,1), (4.3,1) }.

Figure 13 and Figure 14 illustrate the top-down and bottom-up evaluations, respectively, of the EIG tree of participant 2. The following cells are included: $\theta_{1.2}$, $\theta_{1.3}$, $\theta_{1.4}$, θ_1, θ_λ, θ_2, θ_3, θ_4. The top-down evaluation ends after 24 steps, all cells end in the final

state t_z and the final decision value of θ_λ is 0.

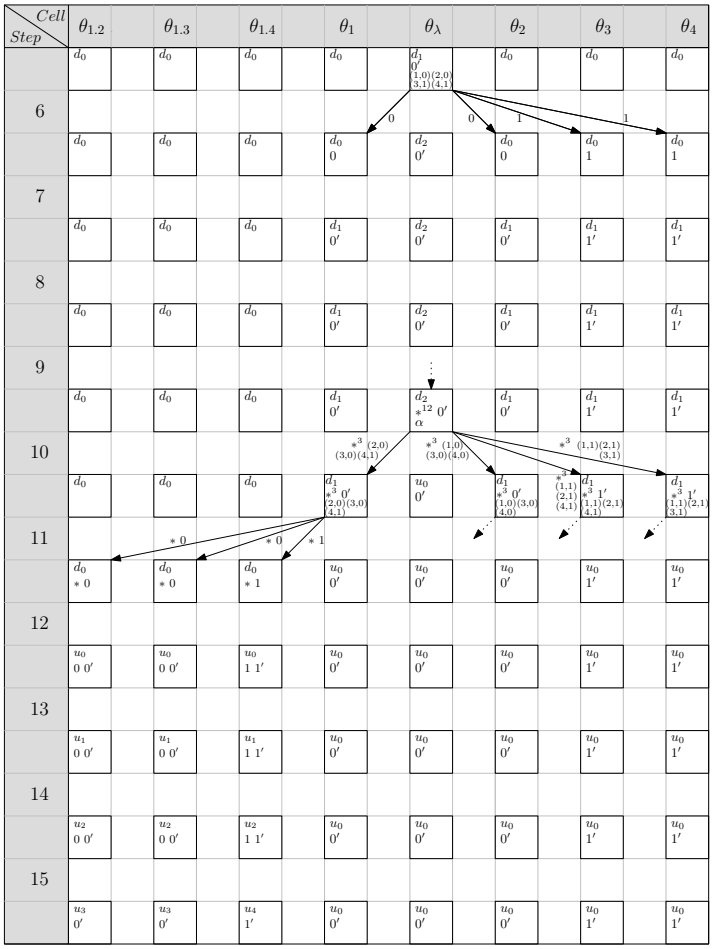

Figure 13: Traces of the top-down evaluation of the EIG tree of participant 2 (fragments). Here, α = { (1.2,0), (1.3,0), (1.4,1), (2.1,0), (2.3,0), (2.4,0), (3.1,1), (3.2,1), (3.4,1), (4.1,1), (4.2,1), (4.3,1) }.

Our P program was also successfully tested on our P system simulator, for a fair number of scenarios, including various combinations of Byzantine behaviours, such as wrong messages, incorrectly formatted messages, extra messages, missing messages, out-of-sync messages.

8 Conclusion

In this paper, we have proposed a new modular framework for designing P system programs and used it to investigate the Byzantine agreement problem. Our modular framework allows encapsulations, information hiding and modular composition. We believe that our solution of the Byzantine agreement problem is the first P system solution for this problem.

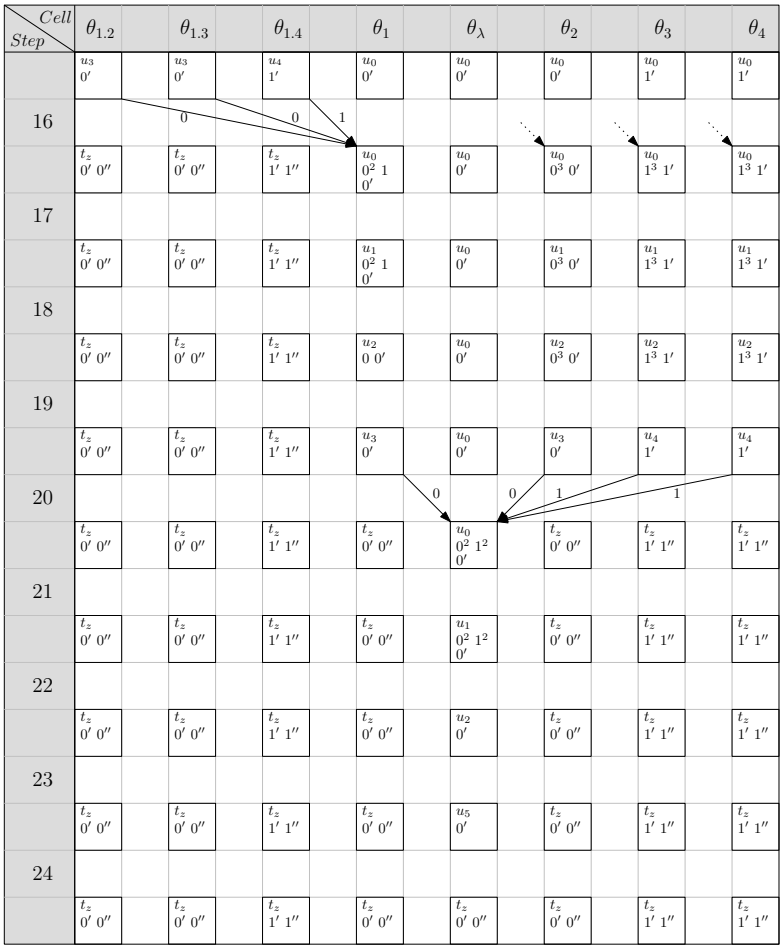

Figure 14: Traces of the bottom-up evaluation of the EIG tree of participant 2 (fragments).

Our investigation leaves open a number of interesting and challenging problems. Can we achieve a Byzantine agreement using only duplex channels (without any simplex channels)? The number of cells and rules of our P program for the Byzantine agreement grows exponentially in N and L and the message size is larger than optimal. Can we reduce the space complexity of our messaging phase? In our program, all the cells must be created and connected before our algorithm starts. Is it possible to solve the same problem with a fixed number of cells? Otherwise, is it possible to solve the same problem starting with a fixed number of cells, and develop a dynamically growing EIG tree? Is it possible to solve the same problem with a fixed number of rules? Can we design P system programs for other Byzantine agreement algorithms, not EIG-based, for example using reliable broadcasts? Can we extend our P system programs to cover $2F + 1$ connected graphs, but not necessarily complete?

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