Gandy-Păun-Rozenberg Machines

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Mathematics is a powerful tool that helps people achieve new goals as well as understand what is impossible to do.

Mark Burgin

Summary. Gandy–Păun–Rozenberg machines are introduced as certain graph rewriting systems. A representation of Gandy–Păun–Rozenberg machines by Gandy machines is given. A construction of a Gandy–Păun–Rozenberg machine solving 3-SAT problem in a polynomial time is shown.

1 Introduction

The paper [8] by Eric Steinhart contains a discussion of logical foundations of computation theory including quantum computing which gives rise to the following family of questions:

(?) what is it an $\mathcal{X}$ possible machine?

for $\mathcal{X} \in \{$set-theoretically, discrete topologically, continuous topologically, geometrically, biologically inspired, physically, cognitive and intelligent$\}$.

We point out here that Robin Gandy's machines (cf. Gandy's paper [1]) yield some answer to (?) for $\mathcal{X} \equiv$ set-theoretically in discrete case. The physically possible machines are discussed in the papers about physical limitations of computing devices by Scott Aaronson, Jacob Bekenstein, Charles H. Bennett, Rolf Landauer, Stockmeyer and Meyer, among others.

The paper [9] by Jiří Wiedermann inspired to formulate (?) for $\mathcal{X} \equiv$ cognitive and intelligent.

An idea of a Gandy–Păun–Rozenberg machine, briefly G–P–R machine, introduced in Section 2, is aimed to provide an answer to (?) for $\mathcal{X} \equiv$ set-theoretically, $\mathcal{X} \equiv$ discrete topologically, and $\mathcal{X} \equiv$ biologically inspired.

The G–P–R machines are the constructs which have common features with or are related to:

— Gandy's machines,
— P systems due to Gheorghe Păun (cf. [6]),
— parallel rewriting systems of graphs investigated by Grzegorz Rozenberg himself with scientists cooperating with him, among others, in preparation and editing of many volume *Handbook of graph grammars and computing by graph transformation* [2].

The core of a G–P–R machine is a finite set of rewriting rules for certain finite directed labelled graphs, where these graphs are instantenous descriptions for the computation process realized by the machine.

The conflictless parallel (simultaneous) application of the rewriting rules of a G–P–R machine is realized in Gandy's machine mode (according to Local Causality Principle), where (local) maximality of "causal neighbourhoods" replaces (global) maximality of, e.g. conflictless set of evolution rules applied simultaneously to a membrane structure which appears during the evolution process generated by a P system. Therefore one can construct a Gandy's machine from a G–P–R machine in an immediate way, see Section 2.

The NP complete problems can be solved by G–P–R machines in a polynomial time (but with an exponential number of indecomposable processors), see Section 3, where we construct a G–P–R machines solving SAT problem in a polynomial time in a similar way to (families of) P systems solving this problem also in a polynomial time (cf. the pioneering Păun's paper [5]).

Randomized G–P–R machines for solving NP problems in a polynomial time with subexponential number of indecomposable processors are forthcoming.

An extension of G–P–R machines to the case of cellular automata can be done by adopting the idea of cellular hypergraph rewriting introduced by Peter Hartmann in his paper [3].

2 Gandy-Păun-Rozenberg machines and Gandy machines

For all unexplained terms and notation of category theory and graph theory we refer the reader to Appendix.

Definition. A G–P–R *machine* $\mathcal{M}$ is determined by the following data:

— a finite set $\Sigma_{\mathcal{M}}$ of labels or symbols of $\mathcal{M}$,
— a skeletal set $\mathcal{S}_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over Σ, which are called *instantenous descriptions* of $\mathcal{M}$,
— a function $\mathcal{F}_{\mathcal{M}} : \mathcal{S}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}}$ called the *transition function* of $\mathcal{M}$,
— a function $\mathcal{R}_{\mathcal{M}}$: PREM$_{\mathcal{M}}$ $\to$ CONCL$_{\mathcal{M}}$ from a finite skeletal set PREM$_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ onto a finite skeletal set CONCL$_{\mathcal{M}}$ of finite isomorphically perfect labelled directed graphs over $\Sigma_{\mathcal{M}}$ such that $\mathcal{R}_{\mathcal{M}}$ determines the *set*

$$\widetilde{\mathcal{R}}_{\mathcal{M}} = \{P \vdash C \mid P \in \mathrm{PREM}_{\mathcal{M}} \text{ and } C = \mathcal{R}_{\mathcal{M}}(P)\}$$

of rewriting rules of $\mathcal{M}$ which are identified with ordered pairs $r = (P_r, C_r)$, where the graph $P_r \in \mathrm{PREM}_{\mathcal{M}}$ is the *premise* of r and the graph $C_r = \mathcal{R}_{\mathcal{M}}(P_r)$ is the *conclusion* of r,

— a subset $\mathcal{I}_{\mathcal{M}}$ of $\mathcal{S}_{\mathcal{M}}$ which is the set of *initial instantaneous descriptions of* $\mathcal{M}$.

The above data are subject of the following conditions:

1) $V(\mathcal{G}) \subseteq V(\mathcal{F}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$,
2) $V(\mathcal{G}) \subseteq V(\mathcal{R}_{\mathcal{M}}(\mathcal{G}))$ for every $\mathcal{G} \in \mathrm{PREM}_{\mathcal{M}}$,
3) the rewriting rules of $\mathcal{M}$ are *applicable* to $\mathcal{S}_{\mathcal{M}}$ which means that for every $\mathcal{G} \in \mathcal{S}_{\mathcal{M}}$ the set

$$\begin{aligned}\mathcal{P}\ell(\mathcal{G}) = \big\{h \mid\ & h \text{ is an embedding of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h) \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h) = \mathcal{G} \\ & \text{such that for every embedding } h' \text{ of labelled graphs over } \Sigma \\ & \text{with } \mathrm{dom}(h') \in \mathrm{PREM}_{\mathcal{M}} \text{ and } \mathrm{cod}(h') = \mathcal{G} \\ & \text{if } \mathrm{im}(h) \text{ is a labelled subgraph of } \mathrm{im}(h'), \text{ then } h = h'\big\}\end{aligned}$$

of *maximal applications* h of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ of $\mathcal{M}$ in places $\mathrm{im}(h)$ is such that the following conditions hold:

(i) $V(\mathcal{G}) = \bigcup\limits_{h \in \mathcal{P}\ell(\mathcal{G})} V(\mathrm{im}(h))$, $E(\mathcal{G}) = \bigcup\limits_{h \in \mathcal{P}\ell(\mathcal{G})} E(\mathrm{im}(h))$,

(ii) for all $h_1, h_2 \in \mathcal{P}\ell(\mathcal{G})$ the equation $\ell_{\mathcal{G}_{h_1}}(\dot{h}_1^{-1}(v)) = \ell_{\mathcal{G}_{h_2}}(\dot{h}_2^{-1}(v))$ holds for every $v \in V(\mathrm{im}(h_1)) \cap V(\mathrm{im}(h_2))$, where $\ell_{\mathcal{G}_{h_1}}$, $\ell_{\mathcal{G}_{h_2}}$ are the labelling functions of $\mathcal{G}_{h_1} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_1))$, $\mathcal{G}_{h_2} = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h_2))$, respectively, and $\dot{h}_1^{-1}$, $\dot{h}_2^{-1}$ are the inverses of isomorphisms induced by the embeddings h_1, h_2, respectively.

(iii) $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is a colimit of a gluing diagram $\mathcal{D}^{\mathcal{G}}$ constructed in the following way (the construction of $\mathcal{D}^{\mathcal{G}}$ is provided by (ii)):

- the set $\mathcal{I}$ of indexes of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{I} = \mathcal{P}\ell(\mathcal{G}) \cup \{\Delta\}$, where $\Delta \notin \mathcal{P}\ell(\mathcal{G})$ is the center of $\mathcal{D}^{\mathcal{G}}$,
- the family $\mathcal{G}_i$ $(i \in \mathcal{I})$ of labelled graphs of $\mathcal{D}^{\mathcal{G}}$ is such that $\mathcal{G}_h = \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ for every $h \in \mathcal{P}\ell(\mathcal{G})$, and $\mathcal{G}_\Delta$ is such that $V(\mathcal{G}_\Delta) = V(\mathcal{G})$, $E(\mathcal{G}_\Delta) = \varnothing$, and the labelling function $\ell_{\mathcal{G}_\Delta}$ is such that provided by (ii)
$$\ell_{\mathcal{G}_\Delta}(v) = \ell_{\mathcal{G}_h}(\dot{h}^{-1}(v))$$
for every $v \in V(\mathrm{im}(h))$ and every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by the embedding h,
- the gluing conditions gl_h $(h \in \mathcal{P}\ell(\mathcal{G}))$ of $\mathcal{D}^{\mathcal{G}}$ are defined by
$$\mathrm{gl}_h = \big\{(v, \dot{h}^{-1}(v)) \mid v \in V(\mathrm{im}(h))\big\}$$
for every $h \in \mathcal{P}\ell(\mathcal{G})$, where $\dot{h}^{-1}$ is the inverse of the isomorphism $\dot{h}$ induced by embedding h,

(iv) the following equations hold:

$$V(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i\in\mathcal{I}} V(\mathrm{im}(q_i))$$
$$\text{and } E(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) = \bigcup_{i\in\mathcal{I}} E(\mathrm{im}(q_i))$$

for the canonical injections $q_i : \mathcal{G}_i \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ $(i \in \mathcal{I})$ forming a colimiting cocone of the diagram $\mathcal{D}^{\mathcal{G}}$ defined in (iii),

(v) the canonical injection $q_{\Delta} : \mathcal{G}_{\Delta} \to \mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is an inclusion of labelled graphs, where Δ is the center of $\mathcal{D}^{\mathcal{G}}$ and q_{Δ} is an element of the colimiting cocone in (iv).

Thus $\mathcal{F}_{\mathcal{M}}(\mathcal{G})$ is the result of simultaneous application of the rules $\mathrm{dom}(h) \vdash \mathcal{R}_{\mathcal{M}}(\mathrm{dom}(h))$ in the places $\mathrm{im}(h)$ for $h \in \mathcal{P}\ell(\mathcal{G})$, where one replaces simultaneously $\mathrm{im}(h)$ by $\mathrm{im}(q_h)$ in $\mathcal{G}$ for $h \in \mathcal{P}\ell(\mathcal{G})$, respectively.

A finite sequence $\big(\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})\big)_{i=0}^{n}$ is called a *finite computation of* $\mathcal{M}$, the number n is called the *time* of this computation, and $\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})$ is called the *final instantaneous description* for this computation if

$$\mathcal{F}^0_{\mathcal{M}}(\mathcal{G}) = \mathcal{G} \in \mathcal{I}_{\mathcal{M}}, \quad \mathcal{F}^{n-1}_{\mathcal{M}}(\mathcal{G}) \neq \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}), \quad \text{and } \mathcal{F}_{\mathcal{M}}(\mathcal{F}^n_{\mathcal{M}}(\mathcal{G})) = \mathcal{F}^n_{\mathcal{M}}(\mathcal{G}),$$

where $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G})$ is defined inductively: $\mathcal{F}^i_{\mathcal{M}}(\mathcal{G}) = \mathcal{F}_{\mathcal{M}}\big(\mathcal{F}^{i-1}_{\mathcal{M}}(\mathcal{G})\big)$. $\square$

We introduce the following auxiliary constructs which will be used to define those Gandy machines which represent G–P–R machines. For all unexplained terms concerning Gandy machines and hereditarily finite sets we refer the reader to [1], [7].

If the set $\Sigma_{\mathcal{M}}$ of labels of a G–P–R machine is an m-element set, we choose a bijection $\nabla : \Sigma_{\mathcal{M}} \to \{1, \ldots, m\}$ and an urelement u to code the labels $\sigma \in \Sigma_{\mathcal{M}}$ by hereditarily finite sets $\{u\}^{\nabla(\sigma)+1}$, where one defines $\{u\}^1 = \{u\}$, $\{u\}^{k+1} = \big\{\{u\}^k\big\}$ for a natural number $k > 0$.

Then for a labelled directed graph $\mathcal{G}$ belonging to the set $\mathcal{S}_{\mathcal{M}}$ of instantaneous descriptions of a G–P–R machine $\mathcal{M}$, an injection $\alpha : V(\mathcal{G}) \to U$ into the set U of urelements, and an urelement $u = \alpha(v)$ for some v one defines a hereditarily finite set

$$H(\alpha, u, \mathcal{G}) = \Big\{\big\{\alpha(v_1), \{\alpha(v_2)\}\big\} \;\Big|\; (v_1, v_2) \in E(\mathcal{G})\Big\}$$
$$\cup \Big\{\big\{\alpha(v), \{\{u\}^{\nabla(\ell_{\mathcal{G}}(v))+1}\}\big\} \;\Big|\; v \in V(\mathcal{G})\Big\},$$

where $\ell_{\mathcal{G}}$ is the labelling function of $\mathcal{G}$.

Since $\mathcal{S}_{\mathcal{M}}$ is a skeletal set of isomorphically perfect graphs, the assignment $H(\alpha, u, \mathcal{G})$ defined above is a bijection from

$$\mathcal{S}^+_{\mathcal{M}} = \big\{(\alpha, u, \mathcal{G}) \mid \mathcal{G} \in \mathcal{S}_{\mathcal{M}},\ \alpha : V(\mathcal{G}) \to U \text{ is an injection,}$$
$$\text{and } u = \alpha(v) \text{ for some } v\big\}$$

into
$$\mathcal{S}^*_{\mathcal{M}} = \{H(\alpha, u, \mathcal{G}) \mid (\alpha, u, \mathcal{G}) \in \mathcal{S}^+_{\mathcal{M}}\},$$
where $\mathcal{S}^*_{\mathcal{M}}$ appears a structural set of hereditarily finite sets understood as in Gandy's paper [1]. We use this set $\mathcal{S}^*_{\mathcal{M}}$ as the set of state-descriptions of a Gandy machine aimed to represent a G–P–R machine $\mathcal{M}$.

Then we choose a mapping $F^+ : \mathcal{S}^+_{\mathcal{M}} \to \mathcal{S}^+_{\mathcal{M}}$ such that
$$F^+(\alpha, u, \mathcal{G}) = (\widehat{\alpha}, u, \mathcal{F}_{\mathcal{M}}(\mathcal{G}))$$
for the transition function $\mathcal{F}_{\mathcal{M}}$ of a G–P–R machine $\mathcal{M}$ and for a chosen injection $\widehat{\alpha} : V(\mathcal{F}_{\mathcal{M}}(\mathcal{G})) \to U$ such that
$$\widehat{\alpha}(v) = \alpha(v) \text{ for every } v \in V(\mathcal{G}) \subseteq V(\mathcal{F}_{\mathcal{M}}(\mathcal{G})).$$
Then we define a mapping $\mathcal{F}^*_{\mathcal{M}} : \mathcal{S}^*_{\mathcal{M}} \to \mathcal{S}^*_{\mathcal{M}}$ such that
$$\mathcal{F}^*_{\mathcal{M}}\big(H(\alpha, u, \mathcal{G})\big) = H\big(F^+(\alpha, u, \mathcal{G})\big).$$
This mapping $\mathcal{F}^*_{\mathcal{M}}$ appears a structural mapping understood as in Gandy's paper [1] and we use it as the transition function of a Gandy machine aimed to represent a G–P–R machine $\mathcal{M}$ which is described in the following theorem.

Theorem 1 (Representation of G–P–R machines by Gandy machines). *Let $\mathcal{M}$ be a G–P–R machine. Then $\mathcal{M}$ determines a Gandy machine $\mathsf{G}_{\mathcal{M}}$ whose set of state-descriptions is $\mathcal{S}^*_{\mathcal{M}}$, the transition function of $\mathsf{G}_{\mathcal{M}}$ is $\mathcal{F}^*_{\mathcal{M}}$, the sets T_1, T_2 of stereotypes of $\mathsf{G}_{\mathcal{M}}$ and the structural functions G_1, G_2 of $\mathsf{G}_{\mathcal{M}}$ are such that*
$$T_1 = T_2 = \mathrm{PREM}^*_{\mathcal{M}}/\cong \quad \textit{and} \quad G_1 = G_2 = \mathcal{R}^*_{\mathcal{M}},$$
where $\mathrm{PREM}^*_{\mathcal{M}}$ is defined for $\mathrm{PREM}_{\mathcal{M}}$ in an analogous way as $\mathcal{S}^*_{\mathcal{M}}$ has been defined for $\mathcal{S}_{\mathcal{M}}$, $\mathrm{PREM}^*_{\mathcal{M}}/\cong$ is the set of equivalence classes with respect to isomorphism relation $\cong$ of hereditarily finite sets defined in Gandy's paper [1]*, and $\mathcal{R}^*_{\mathcal{M}}$ is defined for $\mathcal{R}_{\mathcal{M}}$ in an analogous way as $\mathcal{F}^*_{\mathcal{M}}$ has been defined for $\mathcal{F}_{\mathcal{M}}$.*

Proof. The assumption that $\mathcal{F}(\mathcal{G})$ is a colimit of the gluing diagram $\mathcal{D}^{\mathcal{G}}$ and Lemma 5 in the Appendix provide that the conditions $(3)_r$ of Principle IV in Gandy's paper [1] hold for $\mathsf{G}_{\mathcal{M}}$. □

The assignment $H(\alpha, u, \mathcal{G})$ and then the definition of $\mathcal{S}^*_{\mathcal{M}}$ were inspired by the similar constructions in [7].

The examples of G–P–R machines are presented in the next section.

3 Gandy-Păun-Rozenberg machines and NP complete problems

We show a construction of a G–P–R machine which solves NP complete 3-SAT problem in a polynomial time. We begin with presentation of examples of those

G–P–R machines which simulate the computations of Turing machines and the computations of certain Boolean circuits, respectively, and which are used in the construction.

For all unexplained terms of logic and computational complexity theory, including Turing machines and the formulation of SAT and 3-SAT problems, we refer the reader to [4].

We use the following two types of labelled directed graphs.

Definitions. We say that an ordered triple (k, m, n) of integers k, m, n is *acceptable* if $k > 0$, $m \neq 0$, $n > 1$, and $-k < m < n$. We define

$$\mathrm{lin}[k,n] = \big\{(i, i+1) \mid i \text{ is an integer such that } -k \leq i < -1 \text{ or } 1 \leq i \leq n\big\} \\ \cup \{(-1,1)\}$$

for k, n as above.

Then we say that a labelled directed graph $\mathcal{G}$ over Σ having more than one label is *induced by an acceptable ordered triple* (k, m, n) if $\mathcal{G}$ is such that

— $V(\mathcal{G}) = \big\{i \mid i \text{ is an integer such that } -k \leq i \leq n\big\}$,
— $E(\mathcal{G}) = \mathrm{lin}[k,n] \cup \{(0,m),(1,1)\}$,
— $\ell_{\mathcal{G}}(0) \notin \{\ell_{\mathcal{G}}(k), \ell_{\mathcal{G}}(m)\}$.

For a natural number $n > 0$ a *regular labelled binary tree of depth* n *over* $\{\mathrm{root}, 0, 1\} \times \Sigma$ is defined to be a labelled directed graph $\mathcal{T}$ over $\{\mathrm{root}, 0, 1\} \times \Sigma$ such that

— $V(\mathcal{T})$ is the set of binary strings[1] of length not greater than n including empty string Λ,
— $E(\mathcal{T}) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}) \text{ and } i \in \{0,1\}\big\}$
$\cup \big\{(\Gamma, \Gamma) \mid \Gamma \text{ is a binary string of length } n\big\}$,
— the labelling function $\ell_{\mathcal{T}} : V(\mathcal{T}) \to \{\mathrm{root}, 0, 1\} \times \Sigma$ of $\mathcal{T}$ is such that $\ell^1_{\mathcal{T}}(\Lambda) = \mathrm{root}$, $\ell^1_{\mathcal{T}}(\Gamma i) = i$ for every binary string Γ and every $i \in \{0,1\}$ such that $\Gamma i \in V(\mathcal{T})$,

where $\ell^1_{\mathcal{T}}(x)$, $\ell^2_{\mathcal{T}}(x)$ denote the coordinates such that $\ell_{\mathcal{T}}(x) = (\ell^1_{\mathcal{T}}(x), \ell^2_{\mathcal{T}}(x))$ and Γi denotes that binary string Θ whose last element is the digit i, and Γ is that binary string which is the result of deleting the last element in Θ.

Lemma 1. *The set of labelled directed graphs over Σ induced by acceptable ordered triples of integers is a skeletal set of isomorphically perfect graphs for Σ having more than one label.*

Lemma 2. *The set of all regular binary trees of arbitrary depth over* $\{\mathrm{root}, 0, 1\} \times \Sigma$ *is a skeletal set of isomorphically perfect graphs.*

[1] A binary string is a sequence, maybe empty, of digits 0, 1.

Example 1 (G–P–R machine simulating the computations of a Turing machine). Let $\mathbb{T}$ be a Turing machine whose alphabet Σ (including blank symbol) is disjoint with the set Q of states of $\mathbb{T}$ and let $\delta : \Sigma \times Q \to \Sigma \times Q \times \{L, 0, R\}$ be the transition of $\mathbb{T}$ with cursor directions L for "left", 0 for "stay", and R for "right". We define a *graphical instantaneous description of* $\mathbb{T}$ to be a labelled directed graph $\mathcal{G}$ over $\Sigma^0 = \Sigma \cup Q \cup \{\%, \S\}$ with $\{\%, \S\} \cap (\Sigma \cup Q) = \varnothing$ such that

— $\mathcal{G}$ is induced by some acceptable ordered triple of integers,
— if $\mathcal{G}$ is induced by an acceptable ordered triple (k, m, n) of integers, then $\ell_{\mathcal{G}}(-k) = \%$, $\ell_{\mathcal{G}}(0) \in Q$, $\ell_{\mathcal{G}}(n) = \S$ and $\ell_{\mathcal{G}}(j) \in \Sigma$ for every $j \in \big\{i \in V(\mathcal{G}) \mid -k < i < n \text{ and } i \neq 0\big\}$ (here m corresponds to cursor position on Turing machine tape indicated by the edge $(0, m)$).

By Lemma 1 the set $\mathcal{S}_{\mathbb{T}}$ of all graphical instantaneous descriptions of $\mathbb{T}$ is a skeletal set of isomorphically perfect labelled graphs. Thus we define a G–P–R machine $\mathcal{M}_{\mathbb{T}}$ aimed to simulate the computations of $\mathbb{T}$ such that

— the set of instantaneous descriptions of $\mathcal{M}_{\mathbb{T}}$ is the set $\mathcal{S}_{\mathbb{T}}$ of graphical instantaneous descriptions of $\mathbb{T}$,
— the transition function $\mathcal{F}_{\mathbb{T}}$ of $\mathcal{M}_{\mathbb{T}}$ and the rewriting rules of $\mathcal{M}_{\mathbb{T}}$ are determined by the transition function δ of $\mathbb{T}$ such that if $\delta(a, q) = (a', q', R)$, then
  (f^R) if $\mathcal{G} \in \mathcal{S}_{\mathbb{T}}$ and $\mathcal{G}$ is induced by (k, m, n) such that $\ell_{\mathcal{G}}(m) = a$, $\ell_{\mathcal{G}}(0) = q$, then
    (f^R_1) if $m < n - 1$ then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k, \widehat{m}, n)$ with $\widehat{m} = m + 1$ for $m \neq -1$ and $m = 1$ for $m = -1$ such that $\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for every $i \in V(\mathcal{G}) - \{0, m\}$,
    (f^R_2) if $m = n - 1$ then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k, m+1, n+1)$ such that $\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, $\ell_{\mathcal{G}'}(n)$ is blank symbol, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for every $i \in V(\mathcal{G}') - \{0, m, n\}$,
  (r^R) the rewriting rules are given by the following two schemes $\mathcal{G}_p \vdash \mathcal{G}_c$ such that
    (r^R_1) the premise $\mathcal{G}_p$ is such that $V(\mathcal{G}_p) = \{-1, 0, 1, 2\}$, $E(\mathcal{G}_p) = \mathrm{lin}[1, 2] \cup \{(0, 1)\}$, $\ell_{\mathcal{G}_p}(-1) \in \Sigma \cup \{\%\}$, $\ell_{\mathcal{G}_p}(0) = q$, $\ell_{\mathcal{G}_p}(1) = a$, $\ell_{\mathcal{G}_p}(2) \in \Sigma$,
    the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = V(\mathcal{G}_p)$, $E(\mathcal{G}_c) = \mathrm{lin}[1, 2] \cup \{(0, 2)\}$, $\ell_{\mathcal{G}_c}(-1) = \ell_{\mathcal{G}_p}(-1)$, $\ell_{\mathcal{G}_c}(0) = q'$, $\ell_{\mathcal{G}_c}(1) = a'$, and $\ell_{\mathcal{G}_c}(2) = \ell_{\mathcal{G}_p}(2)$.
    (r^R_2) the premise $\mathcal{G}_p$ is such that $V(\mathcal{G}_p) = \{-1, 0, 1, 2\}$, $E(\mathcal{G}_p) = \mathrm{lin}[1, 2] \cup \{(0, 1)\}$, $\ell_{\mathcal{G}_p}(-1) \in \Sigma \cup \{\%\}$, $\ell_{\mathcal{G}_p}(0) = q$, $\ell_{\mathcal{G}_p}(1) = a$, $\ell_{\mathcal{G}_p}(2) = \S$,
    the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = \{-1, 0, 1, 2, 3\}$, $E(\mathcal{G}_c) = \mathrm{lin}[1, 3] \cup \{(0, 2)\}$, $\ell_{\mathcal{G}_c}(-1) = \ell_{\mathcal{G}_p}(-1)$, $\ell_{\mathcal{G}_c}(0) = q'$, $\ell_{\mathcal{G}_c}(1) = a'$, $\ell_{\mathcal{G}_c}(2)$ is blank symbol, and $\ell_{\mathcal{G}_c}(3) = \S$.

For the cases of equations $\delta(a, q) = (a', q', 0)$ and $\delta(a, q) = (a', q', L)$ the values $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ and the rewriting rules are defined in a similar way, where, e.g., the counterpart of (f^R_2) for $\delta(a, q) = (a', q', L)$ is:

(f^L_2) if $1 = m = k$ or $-k+1 = m \neq 0$, then $\mathcal{F}_{\mathbb{T}}(\mathcal{G})$ is that $\mathcal{G}'$ which is induced by $(k+1, -k, n)$ such that $\ell_{\mathcal{G}'}(-k-1) = \%$, $\ell_{\mathcal{G}'}(-k)$ is blank symbol,

$\ell_{\mathcal{G}'}(0) = q'$, $\ell_{\mathcal{G}'}(m) = a'$, and $\ell_{\mathcal{G}'}(i) = \ell_{\mathcal{G}}(i)$ for all $i \in V(\mathcal{G}') - \{-k-1, -k, 0, m\}$.

The versions of the above rules $\mathcal{G}_p \vdash \mathcal{G}_c$ for both $\mathcal{G}_p$ and $\mathcal{G}_c$ completed by the loop (i,i) for a unique $i \in V(\mathcal{G}_p)$ with $\ell_{\mathcal{G}_p}(i) \notin \{\%, \S\} \cup Q$ are also necessary. The identity rules $\mathcal{G} \vdash \mathcal{G}$ are also necessary, where $\mathcal{G}$ is of the following two forms:

(id$_1$) $V(\mathcal{G}) = \{0,1\}$, $E(\mathcal{G}) = \{(0,1)\}$, $\{\ell_{\mathcal{G}}(0), \ell_{\mathcal{G}}(1)\} \subset \Sigma^0 - Q$,
(id$_2$) $V(\mathcal{G}) = \{0\}$, $E(\mathcal{G}) = \{(0,0)\}$, $\ell_{\mathcal{G}}(0) \in \Sigma$.

There is no other rewriting rule of $\mathcal{M}_{\mathbb{T}}$ than that described by the above schemes.

Since the graphical instantaneous descriptions of a Turing machine $\mathbb{T}$ coincide with the usual instantaneous descriptions of $\mathbb{T}$ or configurations of $\mathbb{T}$ as in [4], the G–P–R machine $\mathcal{M}_{\mathbb{T}}$ simulates the computations of $\mathbb{T}$ due to definition of $\mathcal{F}_{\mathbb{T}}$.

Example 2 (G–P–R machine simulating the computations of certain Boolean circuits). We define a *disjunctive circuit* G–P–R *machine* $\mathcal{M}_{\text{circ}}$ which is aimed to simulate computations of certain tree like Boolean circuits such that

— the set $\mathcal{S}_{\text{circ}}$ of instantaneous descriptions of $\mathcal{M}_{\text{circ}}$ is the set of those regular labelled binary trees $\mathcal{T}$ of depth greater than 3 over the set $\{\text{root}, 0, 1\} \times \{\perp, 0, 1\}$ of labels which satisfy the following condition

(circ$_0$) for every binary string $\Gamma \in V(\mathcal{T})$ of length equal to the depth of $\mathcal{T}$ the number of elements of the set

$$\big\{ i \mid i \text{ is a natural number with } 0 < i \leq n \text{ such that } \ell^2_{\mathcal{T}}(\Gamma \restriction i) \neq \perp \big\}$$

is not greater than 1 (thus this set may be empty), where n is the depth of $\mathcal{T}$ and if Γ is $(k_j)_{j=1}^n$ then $\Gamma \restriction i$ denotes the string $(k_j)_{j=1}^i$ which is Γ itself for $i = n$ and for $i < n$ $(k_j)_{j=1}^i$ is a shortening of Γ by cancellation of the elements $k_n, k_{n-1}, \ldots, k_{i+1}$.

— the transition function $\mathcal{F}_{\text{circ}}$ of $\mathcal{M}_{\text{circ}}$ is such that $\mathcal{F}_{\text{circ}}(\mathcal{T})$ is the result of simultaneous application to $\mathcal{T}$ in G–P–R machine mode the rewriting rules of $\mathcal{M}_{\text{circ}}$ which do not introduce new vertices and which are given by the following three schemes $\mathcal{T}_p \vdash \mathcal{T}_c$ such that

(circ$_1$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 00, 01\}$,
$E(\mathcal{T}_p) = \big\{(\Lambda, 0), (0, 00), (0, 01), (00, 00), (01, 01)\big\}$,
$\ell^2_{\mathcal{T}_p}(\Lambda) = \ell^2_{\mathcal{T}_p}(0) = \perp$, $\{\ell^1_{\mathcal{T}_p}(\Lambda), \ell^1_{\mathcal{T}_p}(0)\} \subseteq \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(00) = 0$, $\ell^1_{\mathcal{T}_p}(01) = 1$, $\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\} \subseteq \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(\mathcal{T}_p)$, $E(\mathcal{T}_c) = E(\mathcal{T}_p)$, $\ell_{\mathcal{T}_c}(\Lambda) = \ell_{\mathcal{T}_p}(\Lambda)$, $\ell_{\mathcal{T}_c}(0) = \big(\ell^1_{\mathcal{T}_p}(0), \max\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\}\big)$,
$\ell_{\mathcal{T}_c}(00) = (0, \perp)$, $\ell_{\mathcal{T}_c}(01) = (1, \perp)$,

(circ$_2$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 00, 01, 000, 001, 010, 011\}$,
$E(\mathcal{T}_p) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}_p) \text{ and } i \in \{0,1\}\big\}$,
$\ell^2_{\mathcal{T}_p}(\Gamma) = \perp$ for all $\Gamma \in V(\mathcal{T}_p) - \{00, 01\}$,

$\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\} \subseteq \{0,1\}$, $\{\ell^1_{\mathcal{T}_p}(\Lambda), \ell^1_{\mathcal{T}_p}(0)\} \subseteq \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(\Gamma i) = i$ for all $\Gamma \in \{0, 00, 01\}$ and $i \in \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(\mathcal{T}_p)$, $E(\mathcal{T}_c) = E(\mathcal{T}_p)$, $\ell_{\mathcal{T}_c}(\Gamma) = \ell_{\mathcal{T}_p}(\Gamma)$ for every $\Gamma \in V(\mathcal{T}_c) - \{0, 00, 01\}$,
$\ell_{\mathcal{T}_c}(0) = \big(\ell_{\mathcal{T}_p}(0), \max\{\ell^2_{\mathcal{T}_p}(00), \ell^2_{\mathcal{T}_p}(01)\}\big)$,
$\ell_{\mathcal{T}_c}(\Gamma) = \big(\ell^1_{\mathcal{T}_p}(\Gamma), \perp\big)$ for every $\Gamma \in \{00, 11\}$,

(circ$_3$) the premise $\mathcal{T}_p$ is such that $V(\mathcal{T}_p) = \{\Lambda, 0, 1, 00, 01, 10, 11\}$,
$E(\mathcal{T}_p) = \big\{(\Gamma, \Gamma i) \mid \{\Gamma, \Gamma i\} \subseteq V(\mathcal{T}_p) \text{ and } i \in \{0,1\}\big\}$,
$\ell^1_{\mathcal{T}_p}(\Gamma i) = i$ for all $\Gamma \in \{\Lambda, 0, 1\}$ and $i \in \{0,1\}$,
$\ell^2_{\mathcal{T}_p}(\Gamma) = \perp$ for every $\Gamma \in V(\mathcal{T}_p) - \{0,1\}$,
$\ell^1_{\mathcal{T}_p}(\Lambda) = \text{root}$, $\{\ell^2_{\mathcal{T}_p}(0), \ell^2_{\mathcal{T}_p}(1)\} \subseteq \{0,1\}$,
the conclusion $\mathcal{T}_c$ is such that $V(\mathcal{T}_c) = V(T_p)$, $E(T_c) = E(\mathcal{T}_p)$,
$\ell^2_{\mathcal{T}_c}(\Gamma) = \ell^2_{\mathcal{T}_p}(\Gamma)$ for every $\Gamma \in V(\mathcal{T}_c) - \{\Lambda, 0, 1\}$,
$\ell_{\mathcal{T}_c}(\Gamma) = (\ell^1_{\mathcal{T}_p}, \perp)$ for every $\Gamma \in \{0,1\}$,
and $\ell_{\mathcal{T}_c}(\Lambda) = \big(\text{root}, \max\{\ell_{\mathcal{T}_p}(0), \ell_{\mathcal{T}_p}(1)\}\big)$.

The identity rules $\mathcal{T} \vdash \mathcal{T}$ are also necessary which are defined in a similar way as in Example 1.

There is no other rewriting rule of $\mathcal{M}_{\text{circ}}$ than that described by the above schemes.

The following lemma characterizes the computations of G–P–R machine $\mathcal{M}_{\text{circ}}$.

Lemma 3. *Let $\mathcal{T} \in \mathcal{S}_{\text{circ}}$ be a regular labelled binary tree such that the following conditions hold*

(a) *for every binary string Γ of length equal to the depth of $\mathcal{T}$ there exists a natural number i with $i > 0$ such that $\ell^2_{\mathcal{T}}(\Gamma \upharpoonright i) \neq \perp$,*
(b) *for every $\Gamma \in V(\mathcal{T})$ and $j \in \{0,1\}$ if $\Gamma j \in V(\mathcal{T})$ and $\ell^2_{\mathcal{T}}(\Gamma j) \neq \perp$, then $\ell^2_{\mathcal{T}}(\Gamma \neg(j)) \neq \perp$, where $\neg(0) = 1$ and $\neg(1) = 0$.*

Then for

$$n = \max\big\{i \mid i \text{ is the length of some binary string } \Gamma \in V(\mathcal{T}) \text{ with } \ell^2_{\mathcal{T}}(\Gamma) \neq \perp\big\}$$

the value $\mathcal{F}^n_{\text{circ}}(\mathcal{T})$ is that regular labelled tree $\mathcal{T}'$ which is such that $V(\mathcal{T}') = V(\mathcal{T})$, $E(\mathcal{T}') = E(\mathcal{T})$, $\ell^2_{\mathcal{T}'}(\Gamma) = \perp$ for all $\Gamma \in V(\mathcal{T}') - \{\Lambda\}$ and $\ell^2_{\mathcal{T}'}(\Lambda) = \max\big\{\ell^2_{\mathcal{T}}(\Gamma) \mid \Gamma \in V(\mathcal{T}') \text{ and } \ell^2_{\mathcal{T}}(\Gamma) \neq \perp\big\}$, where $\mathcal{F}^n_{\text{circ}}(\mathcal{T})$ is defined inductively by $\mathcal{F}^1_{\text{circ}}(\mathcal{T}) = \mathcal{F}_{\text{circ}}(\mathcal{T})$ and $\mathcal{F}^n_{\text{circ}}(\mathcal{T}) = \mathcal{F}_{\text{circ}}(\mathcal{F}^{n-1}_{\text{circ}}(\mathcal{T}))$.

Example 3 (A G–P–R machine solving 3-SAT problem in a polynomial time). We use a Turing machine $\dot{\mathbb{T}}$ such that for every formula φ in a disjunctive normal form as in 3-SAT problem and every truth assignment T for variables of φ the machine decides in the time $\leq n^{k_0}$ whether φ is valid for T, where the ordered pair (φ, T) is an input for $\dot{\mathbb{T}}$ from which the machine begins the computation, k_0 is some constant natural number, and n is the number of variables occurring in φ. We claim for $\dot{\mathbb{T}}$ that:

(A) if n is the number of variables occurring in φ, then any truth assignment T for variables of φ is represented by that binary string Γ of length n in the machine tape which is such that if the value "True" is assigned to the i-th variable of φ, then 1 is the i-th element of Γ, otherwise the i-th element of Γ is 0,

(B) for the G–P–R machine $\mathcal{M}_{\dot{\mathbb{T}}}$ simulating the computations of $\dot{\mathbb{T}}$ if we have that

  (b_1) $\mathcal{G}_{\varphi,\Gamma}$ is that instantaneous description of $\mathcal{M}_{\dot{\mathbb{T}}}$ which coincides with the initial instantaneous description or initial configuration for input (φ, T) with T represented by the binary string Γ in the machine tape as in (A),

  (b_2) $\mathcal{G} = \mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ is the final instantaneous description of $\mathcal{M}_{\dot{\mathbb{T}}}$ for the case of the final or halting state "stop" reached by $\dot{\mathbb{T}}$ after q steps of computation starting with input (φ, T) with T related to Γ as in (A),

then $\mathcal{G}$ is a labelled graph induced by some acceptable triple (k, m, n) of integers with $(-k, m) \in E(\mathcal{G})$ such that $\ell_{\mathcal{G}}(0)$ is the final state "stop" and $\ell_{\mathcal{G}}(m) = 1$ if φ is valid for the truth assignment represented by Γ, otherwise $\ell_{\mathcal{G}}(m) = 0$, where $\mathcal{F}_{\dot{\mathbb{T}}}$ is the transition function of $\mathcal{M}_{\dot{\mathbb{T}}}$ and $\mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ is inductively defined: $\mathcal{F}^1_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}) = \mathcal{F}_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ and $\mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}) = \mathcal{F}_{\dot{\mathbb{T}}}(\mathcal{F}^{q-1}_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma}))$.

The shape of formulas in a disjunctive normal form in 3-SAT problem (it suffices to consider formulas of $n > 3$ variables which are disjunctions of $2^3 \cdot \binom{n}{3}$ nonrepetitive clauses, each conjunction of three literals containing different variables) provides that the claimed Turing machine $\dot{\mathbb{T}}$ can be constructed from some simpler three-string or three-tape Turing machine 3-$\mathbb{T}$ according to the general construction in the proof of Theorem 2.1, p. 30 of [4]. The first tape of 3-$\mathbb{T}$ is an input tape containing some presentation of a formula, the second tape is also an input tape containing some presentation of a truth assignment, and the third tape is an output tape. The machine 3-$\mathbb{T}$ reads only its input tapes and does not move its head or cursor on output tape printing or erasing the digits 0, 1. The machine 3-$\mathbb{T}$ reaches the final state in the time not greater than $2^3 \cdot n^5$ steps for a formula of n variables, hence by Theorem 2.1, p. 30 of [4] the machine $\dot{\mathbb{T}}$ reaches the final state in the time not greater than $2^6 \cdot n^{10}$ steps for a formula of n variables.

We outline a construction of a G–P–R machine $\mathcal{M}_{\text{3-SAT}}$ aimed to solve 3-SAT problem in a polynomial time. The initial instantaneous descriptions of $\mathcal{M}_{\text{3-SAT}}$ are labelled directed graphs $\mathcal{G}^0_\varphi$ determined by formulas φ in disjunctive normal forms as in 3-SAT problem in the following way:

(I_V) $V(\mathcal{G}^0_\varphi) = \{(j, \Gamma) \mid j \in V(\mathcal{G}_{\varphi,\Gamma}) \text{ and } \Gamma \in 2^n\} \cup \{(\Theta, \Lambda) \mid \Theta \in V(\mathcal{T}_\perp)\}$,
where $\mathcal{G}_{\varphi,\Gamma}$ is that initial instantaneous description which was introduced in (b_1), n is the number of variables occurring in φ, 2^n is the set of binary strings of length n, and $\mathcal{T}_\perp$ is the regular labelled binary tree of depth $n-1$ such that $\ell^2_{\mathcal{T}_\perp}(\Theta) = \perp$ for every $\Theta \in V(\mathcal{T}_\perp)$,

(I_E) $E(\mathcal{G}^0_\varphi) = \{((j, \Gamma), (j', \Gamma)) \mid (j, j') \in E(\mathcal{G}_{\varphi,\Gamma}) \text{ and } \Gamma \in 2^n\}$
$\cup \{((\Theta, \Lambda), (\Theta', \Lambda)) \mid (\Theta, \Theta') \in E(\mathcal{T}_\perp)\}$

$\cup\big\{\big((\Theta,\Lambda),(j_{\Theta i},\Theta i)\big) \mid \Theta i \in 2^n,\ i\in\{0,1\},\ \text{and } \Theta \in 2^{n-1}\big\}$,
where $j_{\Theta i}$ is that unique vertex of $\mathcal{G}_{\varphi,\Theta i}$ for which $\ell_{\mathcal{G}_{\varphi,\Theta i}}(j_{\Theta i}) = \%$,

(I_ℓ) the labelling function $\ell_{\mathcal{G}^0_\varphi}$ is defined in the following way:

- $\ell_{\mathcal{G}^0_\varphi}((j,\Gamma)) = \ell_{\mathcal{G}_{\varphi,\Gamma}}(j)$ for all $\Gamma \in 2^n$ and $j \in V(\mathcal{G}_{\varphi,\Gamma})$ except j_Γ for which $\ell_{\mathcal{G}_{\varphi,\Gamma}}(j_\Gamma) = \%$,
- $\ell_{\mathcal{G}^0_\varphi}((j_\Gamma,\Gamma)) = (i,\%)$ if Γ is Θi for $i \in \{0,1\}$, where j_Γ is such that $\ell_{\mathcal{G}_{\varphi,\Gamma}}(j_\Gamma) = \%$,
- $\ell_{\mathcal{G}^0_\varphi}((\Theta,\Lambda)) = \ell_{\mathcal{T}_\perp}(\Theta)$ for every $\Theta \in V(\mathcal{T}_\perp)$.

Then we define inductively the labelled graphs $\mathcal{G}^k_\varphi$ for a natural number $k > 0$ and a formula φ in a disjunctive normal form as in 3-SAT problem such that the sets $V(\mathcal{G}^k_\varphi)$ and $E(\mathcal{G}^k_\varphi)$ are defined in an analogous way as $V(\mathcal{G}^0_\varphi)$ and $E(\mathcal{G}^0_\varphi)$ were defined in (I_V) and (I_E), respectively, except the graphs $\mathcal{G}_{\varphi,\Gamma}$ are replaced by $\mathcal{F}^k_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ (see the definition of $\mathcal{F}^q_{\dot{\mathbb{T}}}(\mathcal{G}_{\varphi,\Gamma})$ in (B)). The labelling function $\ell_{\mathcal{G}^k_\varphi}$ of $\mathcal{G}^k_\varphi$ is determined by the labelling function of $\mathcal{G}^{k-1}_\varphi$ by imposing that $\mathcal{G}^k_\varphi$ is the result of simultaneous application to $\mathcal{G}^{k-1}_\varphi$ in G–P–R machine mode the rules of the G–P–R machines $\mathcal{M}_{\dot{\mathbb{T}}}$ and $\mathcal{M}_{\mathrm{circ}}$ with the label % replaced by $(i,\%)$ for $i \in \{0,1\}$, and the following new rules given by the scheme $\mathcal{G}_p \vdash \mathcal{G}_c$, where the premise $\mathcal{G}_p$ is such that

$V(\mathcal{G}_p) = \{\Lambda, 0, 00, 01, 001, 011, 0010, 0011, 0110, 0111\}$,
$E(\mathcal{G}_p) = \big\{(\Gamma,\Gamma i) \mid \{\Gamma,\Gamma i\} \subseteq V(\mathcal{G}_p) - \{0010, 0110\} \text{ and } i \in \{0,1\}\big\}$
$\cup\,\{(0010, 001), (0110, 011), (0, 0)\}$,
$\ell^2_{\mathcal{G}_p}(\Lambda) = \ell^2_{\mathcal{G}_p}(0) = \perp$ [2], $\{\ell^1_{\mathcal{G}_p}(\Lambda), \ell^1_{\mathcal{G}_p}(0)\} \subseteq \{0,1\}$,
$\ell_{\mathcal{G}_p}(00) = (0,\%)$, $\ell_{\mathcal{G}_p}(01) = (1,\%)$, $\{\ell_{\mathcal{G}_p}(001), \ell_{\mathcal{G}_p}(011)\} \subseteq \{0,1\}$,
$\ell_{\mathcal{G}_p}(0010) = \ell_{\mathcal{G}_p}(0110) =$ “stop” $\in Q$, $\{\ell_{\mathcal{G}_p}(0011), \ell_{\mathcal{G}_p}(0111)\} \subseteq \Sigma$,

the conclusion $\mathcal{G}_c$ is such that $V(\mathcal{G}_c) = V(\mathcal{G}_p)$, $E(\mathcal{G}_c) = E(\mathcal{G}_p)$,
$\ell_{\mathcal{G}_c}(v) = \ell_{\mathcal{G}_p}(v)$ for every $v \in V(\mathcal{G}_p)$ except $\ell_{\mathcal{G}_c}(001) = \ell_{\mathcal{G}_c}(011) = \perp$,
and $\ell^2_{\mathcal{G}_c}(0) = \max\{\ell_{\mathcal{G}_p}(001), \ell_{\mathcal{G}_p}(011)\}$, $\ell^1_{\mathcal{G}_c}(0) = \ell^1_{\mathcal{G}_p}(0)$.

Thus we define the set $\mathcal{S}_{\text{3-SAT}}$ of instantaneous descriptions of G–P–R machine $\mathcal{M}_{\text{3-SAT}}$ by

$$\mathcal{S}_{\text{3-SAT}} = \big\{\mathcal{G}^k_\varphi \mid k \text{ is a natural number and } \varphi \text{ is a formula in a disjunctive normal form as in 3-SAT problem}\big\}.$$

The transition function $\mathcal{F}_{\text{3-SAT}}$ of G–P–R machine $\mathcal{M}_{\text{3-SAT}}$ is given by

$$\mathcal{F}_{\text{3-SAT}}(\mathcal{G}^k_\varphi) = \mathcal{G}^{k+1}_\varphi \text{ for every } k \geq 0 \text{ and every } \varphi.$$

The rewriting rules of $\mathcal{M}_{\text{3-SAT}}$ are the rewriting rules of the G–P–R machines $\mathcal{M}_{\dot{\mathbb{T}}}$, $\mathcal{M}_{\mathrm{circ}}$ with the label % replaced by $(i,\%)$ for $i \in \{0,1\}$, and the new rules introduced above.

Theorem 2. *The* G–P–R *machine* $\mathcal{M}_{\textit{3-SAT}}$ *solves* 3-SAT *problem in a polynomial time.*

[2] We assume that $\perp \notin \Sigma \cup Q \cup \{\%, \S\}$.

Proof. Since the upper bound of the time of computation of $\dot{\mathbb{T}}$ does not depend on the binary sequences Γ representing truth assignments but only on their length and is polynomial with respect to this length, the upper bound of the time of computation of G–P–R machine $\mathcal{M}_{\dot{\mathbb{T}}}$ also does not depend on binary sequences Γ representing truth assignments and is polynomial with respect to the length of these sequences Γ. Hence by Lemma 3 we get the theorem.

Less formally, for a given formula φ of n variables with $n > 3$ the machine $\mathcal{M}_{\text{3-SAT}}$ simultaneously simulates without any delay the computations of 2^n copies of the Turing machine $\dot{\mathbb{T}}$, where 2^n possible truth assignments for φ are the inputs together with φ itself for these 2^n copies of $\dot{\mathbb{T}}$, respectively. Here each truth assignment T is associated to that copy of $\dot{\mathbb{T}}$ which is aimed to decide whether φ is valid for T.

Then Boolean circuit part of $\mathcal{M}_{\text{3-SAT}}$ simulates the computation of tree-like Boolean circuit $\mathcal{C}$ of 2^n input gates, where the underlying graph of $\mathcal{C}$ is a tree of depth n and all non-input gates of $\mathcal{C}$ are OR gates. The 2^n input gates of $\mathcal{C}$ receive those inputs which are the output results of the computations of the above 2^n copies of $\dot{\mathbb{T}}$, respectively. Here each input gate g is associated with that copy $\dot{\mathbb{T}}_g$ of $\dot{\mathbb{T}}$ for which g is connected with that unique vertex i of the final graphical instantaneous description of $\dot{\mathbb{T}}_g$ for which $(0, i)$ is an edge of this final graphical instantaneous description and i is labelled by the output result of $\dot{\mathbb{T}}_g$ with 0 labelled by the final or halting state of $\dot{\mathbb{T}}_g$. The inputs of $\mathcal{C}$ are simultaneously processed by $\mathcal{C}$ to give the output result in the root of the underlying graph of $\mathcal{C}$. The output result contained in the root yields an answer to a question whether there exists a truth assignment for φ such that φ is valid for this assignment. Therefore $\mathcal{M}_{\text{3-SAT}}$ solves 3-SAT problem in a polynomial time. □

Corollary. *There exists a Gandy machine which solves* 3-SAT *problem in a polynomial time but with the exponential number of urelement processors.*

Proof. The corollary is a consequence of Theorems 1 and 2. □

4 Concluding remarks

One could adopt G–P–R machines and Gandy machines as the underlying abstract computing devices of computational complexity theory because these machines propose a wide scope of possible computational parallelism, even up to unreliable parallelism of G–P–R machine $\mathcal{M}_{\text{3-SAT}}$ and representing it Gandy machine which prove that polynomial computational time does not imply polynomial computational space understood as the size of hardware measured by the number of urelement (indecomposable) processors of a machine. We will show in a forthcoming paper about randomized G–P–R machines those G–P–R machines which are capable to construct in polynomial time the initial instantaneous descriptions of the machine $\mathcal{M}_{\text{3-SAT}}$ from simpler labelled graphs of size, i.e., the number of vertices, depending linearly on the size of input data of a formula and a truth assignment, where some versions of division rules of membrane computing [6] are used.

Appendix. Graph-theoretical and category-theoretical preliminaries

A [*finite*] *labelled directed graph over* a set Σ of labels is defined to be an ordered triple $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), \ell_{\mathcal{G}})$, where $V(\mathcal{G})$ is a [finite] *set of vertices* of $\mathcal{G}$, $E(\mathcal{G})$ is a subset of $V(\mathcal{G}) \times V(\mathcal{G})$ called the *set of edges* of $\mathcal{G}$, and $\ell_{\mathcal{G}}$ is a function from $V(\mathcal{G})$ into Σ called the *labelling function* of $\mathcal{G}$. We drop the adjective 'directed' if there is no risk of confusion.

A *homomorphism of a labelled directed graph $\mathcal{G}$ over Σ into a labelled directed graph $\mathcal{G}'$ over Σ* is an ordered triple $(\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ such that h is a function from $V(\mathcal{G})$ into $V(\mathcal{G}')$ which satisfies the following conditions:

(H_1) $(v, v') \in E(\mathcal{G})$ implies $(\mathrm{h}(v), \mathrm{h}(v')) \in E(\mathcal{G}')$ for all $v, v' \in V(\mathcal{G})$,
(H_2) $\ell_{\mathcal{G}'}(\mathrm{h}(v)) = \ell_{\mathcal{G}}(v)$ for every $v \in V(\mathcal{G})$.

If a triple $h = (\mathcal{G}, \mathrm{h} : V(\mathcal{G}) \to V(\mathcal{G}'), \mathcal{G}')$ is a homomorphism of a labelled directed graph $\mathcal{G}$ over Σ into a labelled directed graph $\mathcal{G}'$ over Σ, we denote this triple by $h : \mathcal{G} \to \mathcal{G}'$, we write $\mathrm{dom}(h)$ and $\mathrm{cod}(h)$ for $\mathcal{G}$ and $\mathcal{G}'$, respectively, according to category theory convention, and we write $h(v)$ for the value $\mathrm{h}(v)$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *embedding of $\mathcal{G}$ into $\mathcal{G}'$*, denoted by $h : \mathcal{G} \rightarrowtail \mathcal{G}'$, if the following condition holds:

(E)$h(v) = h(v')$ implies $v = v'$ for all $v, v' \in V(\mathcal{G})$.

An embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ is an *inclusion of $\mathcal{G}$ into $\mathcal{G}'$*, denoted by $h : \mathcal{G} \hookrightarrow \mathcal{G}'$, if the following holds:

(I) $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is a *labelled subgraph* of a labelled directed graph $\mathcal{G}'$ over Σ if there exists an inclusion $h : \mathcal{G} \hookrightarrow \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define the *image* of h, denoted by $\mathrm{im}(h)$, to be a labelled directed graph $\widehat{\mathcal{G}}$ over Σ such that $V(\widehat{\mathcal{G}}) = \big\{h(v) \mid v \in V(\mathcal{G})\big\}$, $E(\widehat{\mathcal{G}}) = \big\{(h(v), h(v')) \mid (v, v') \in E(\mathcal{G})\big\}$, and the labelling function $\ell_{\hat{\mathcal{G}}}$ of $\widehat{\mathcal{G}}$ is the restriction of the labelling function $\ell_{\mathcal{G}'}$ of $V(\mathcal{G}')$ to the set $V(\widehat{\mathcal{G}})$, i.e., $\ell_{\hat{\mathcal{G}}}(v) = \ell_{\mathcal{G}'}(v)$ for every $v \in V(\widehat{\mathcal{G}})$.

A homomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled directed graphs over Σ is an *isomorphism* of $\mathcal{G}$ into $\mathcal{G}'$ if there exists a homomorphism $h^{-1} : \mathcal{G}' \to \mathcal{G}$ of labelled directed graphs over Σ, called the inverse of h, such that the following conditons hold:

(Iz_1) $h^{-1}(h(v)) = v$ for every $v \in V(\mathcal{G})$, (Iz$_2$) $h(h^{-1}(v)) = v$ for every $v \in V(\mathcal{G}')$.

We say that a labelled directed graph $\mathcal{G}$ over Σ *is isomorphic* to a labelled directed graph $\mathcal{G}'$ over Σ if there exists an isomorphism $h : \mathcal{G} \to \mathcal{G}'$ of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ.

For an embedding $h : \mathcal{G} \rightarrowtail \mathcal{G}'$ of labelled directed graphs $\mathcal{G}, \mathcal{G}'$ over Σ we define a homomorphism $\dot{h} : \mathcal{G} \to \mathrm{im}(h)$ by $\dot{h}(v) = h(v)$ for every $v \in V(\mathcal{G})$. This homomorphism $\dot{h}$ is an isomorphism of $\mathcal{G}$ into $\mathrm{im}(h)$, called an *isomorphism deduced by h*.

For a labelled directed graph $\mathcal{G}$ over Σ, the *identity homomorphism* (or simply, *identity of $\mathcal{G}$*), denoted by $\mathrm{id}_{\mathcal{G}}$, is the homomorphism $h : \mathcal{G} \to \mathcal{G}$ such that $h(v) = v$ for every $v \in V(\mathcal{G})$.

We say that a labelled directed graph $\mathcal{G}$ over Σ is an *isomorphically perfect* labelled directed graph over Σ if the identity homomorphism $\mathrm{id}_{\mathcal{G}}$ is a unique isomorphism of labelled directed graph $\mathcal{G}$ into $\mathcal{G}$.

Lemma 4. *Let $\mathcal{G}$ be an isomorphically perfect labelled directed graph over Σ and let $h : \mathcal{G} \to \mathcal{G}'$, $h' : \mathcal{G} \to \mathcal{G}'$ be two isomorphisms of labelled graphs $\mathcal{G}, \mathcal{G}'$ over Σ. Then $h = h'$.*

We say that a set or a class $\mathcal{A}$ of labelled directed graphs over Σ is *skeletal* if for all labelled directed graphs $\mathcal{G}, \mathcal{G}'$ in $\mathcal{A}$ if they are isomorphic, then $\mathcal{G} = \mathcal{G}'$.

A *gluing diagram* $\mathcal{D}$ of labelled directed graphs over Σ is defined by:

— its *set $\mathcal{I}$ of indexes* with a distinguished index $\Delta \in \mathcal{I}$, called the *center* of $\mathcal{D}$,
— its *family* $\mathcal{G}_i$ $(i \in \mathcal{I})$ *of labelled directed graphs* over Σ,
— its *family* gl_i $(i \in \mathcal{I} - \{\Delta\})$ *gluing conditions* which are sets of ordered pairs such that
  (i) $\mathrm{gl}_i \subseteq V(\mathcal{G}_\Delta) \times V(\mathcal{G}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,
  (ii) $(v, v') \in \mathrm{gl}_i$ implies $\ell_{\mathcal{G}_\Delta}(v) = \ell_{\mathcal{G}_i}(v)$ for all $v \in V(\mathcal{G}_\Delta)$, $v' \in V(\mathcal{G}_i)$, and for every $i \in \mathcal{I} - \{\Delta\}$,
  (iii) for every $i \in \mathcal{I} - \{\Delta\}$ if gl_i is non-empty, then there exists a bijection

$$b_i : L(\mathrm{gl}_i) \to R(\mathrm{gl}_i)$$

for $L(\mathrm{gl}_i) = \{v \mid (v, v') \in \mathrm{gl}_i \text{ for some } v'\}$ and $R(\mathrm{gl}_i) = \{v' \mid (v, v') \in \mathrm{gl}_i \text{ for some } v\}$ such that $\{(v, b_i(v)) \mid v \in L(\mathrm{gl}_i)\} = \mathrm{gl}_i$.

For a gluing diagram $\mathcal{D}$ of labelled directed graphs over Σ we define a *cocone* of $\mathcal{D}$ to be a family $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of homomorphisms of labelled directed graphs over Σ (here $\mathrm{cod}(h_i) = \mathcal{G}$ for every $i \in \mathcal{I}$) such that

$$l_{\mathcal{G}}(h_\Delta(v)) = l_{\mathcal{G}}(h_i(v'))$$

for every pair $(v, v') \in \mathrm{gl}_i$ and every $i \in \mathcal{I} - \{\Delta\}$.

A cocone $q_i : \mathcal{G}_i \to \widetilde{\mathcal{G}}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ is called a *colimiting cocone of $\mathcal{D}$* if for every cocone $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ there exists a unique homomorphism

$h : \widetilde{\mathcal{G}} \to \mathcal{G}$ of labelled directed graphs $\widetilde{\mathcal{G}}, \mathcal{G}$ over Σ such that $h(q_i(v)) = h_i(v)$ for every $v \in V(\mathcal{G}_i)$ and for every $i \in \mathcal{I}$. The labelled directed graph $\widetilde{\mathcal{G}}$ is called a *colimit* of $\mathcal{D}$, the homomorphisms q_i $(i \in \mathcal{I})$ are called *canonical injections* and the unique homomorphism h is called the *mediating morphism* for $h_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$.

For a gluing diagram $\mathcal{D}$ one constructs its colimit $\widetilde{\mathcal{G}}$ in the following way:

— $V(\widetilde{\mathcal{G}}) = \bigcup_{i \in \mathcal{I}} (V_i \times \{i\})$, where
$V_\Delta = V(\mathcal{G}_\Delta)$ for the center Δ of $\mathcal{D}$,
$V_i = V(\mathcal{G}_i) - R(\mathrm{gl}_i)$ for every $i \in \mathcal{I} - \{\Delta\}$,

— $E(\widetilde{\mathcal{G}}) = \bigcup_{i \in \mathcal{I}} E_i$, where

$$E_\Delta = \big\{\big((v,\Delta),(v',\Delta)\big) \mid (v,v') \in E(\mathcal{G}_\Delta)\big\} \text{ for the center } \Delta \text{ of } \mathcal{D},$$

$$\begin{aligned} E_i = {} & \big\{\big((v,i),(v',i)\big) \mid (v,v') \in E(\mathcal{G}_i) \text{ and } \{v,v'\} \subseteq V_i\big\} \\ & \cup \big\{\big((v,\Delta),(v',\Delta)\big) \mid (v,v'') \in \mathrm{gl}_i,\ (v',v''') \in \mathrm{gl}_i, \\ & \qquad \text{and } (v'',v''') \in E(\mathcal{G}_i) \text{ for some } v'',v'''\big\} \\ & \cup \big\{\big((v,\Delta),(v',i)\big) \mid v' \in V_i,\ (v,v'') \in \mathrm{gl}_i \text{ and } (v'',v') \in E(\mathcal{G}_i) \text{ for some } v''\big\} \\ & \cup \big\{\big((v,i),(v',\Delta)\big) \mid v \in V_i,\ (v,v'') \in \mathrm{gl}_i \text{ and } (v,v'') \in E(\mathcal{G}_i) \text{ for some } v''\big\} \end{aligned}$$

for every $i \in \mathcal{I} - \{\Delta\}$,

— the labelling function $\ell_{\widetilde{\mathcal{G}}}$ is defined by $\ell_{\widetilde{\mathcal{G}}}((v,i)) = \ell_{\mathcal{G}_i}(v)$ for every $(v,i) \in V(\widetilde{\mathcal{G}})$.

The definition of a colimiting cocone of a gluing diagram $\mathcal{D}$ provides that any other colimit of $\mathcal{D}$ is isomorphic to the colimit of $\mathcal{D}$ constructed above. Hence one proves the following lemma.

Lemma 5. *Let $\mathcal{D}$ be a gluing diagram of labelled graphs over Σ. Then for every colimiting cocone $q_i : \mathcal{G}_i \to \mathcal{G}$ $(i \in \mathcal{I})$ of $\mathcal{D}$ if $i' \neq i''$, then*

$$\big(V(\mathrm{im}(q_{i'})) - V(\mathrm{im}(q_\Delta))\big) \cap \big(V(\mathrm{im}(q_{i''})) - V(\mathrm{im}(q_\Delta))\big) = \varnothing$$

for all $i', i'' \in \mathcal{I} - \{\Delta\}$, where Δ is the center of $\mathcal{D}$ and the elements of nonempty $V(\mathrm{im}(q_i)) - V(\mathrm{im}(q_\Delta))$ with $i \neq \Delta$ are 'new' elements and the elements of $V(\mathrm{im}(q_\Delta))$ are 'old' elements.

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