# Characterizing the Aperiodicity of Irreducible Markov Chains by Using P Systems

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**Summary.** It is well known that any irreducible and aperiodic Markov chain has exactly one stationary distribution, and for any arbitrary initial distribution, the sequence of distributions at time $n$ converges to the stationary distribution, that is, the Markov chain is approaching equilibrium as $n \to \infty$.

In this paper, a characterization of the aperiodicity in existential terms of some state is given. At the same time, a P system with external output is associated with any irreducible Markov chain. The designed system provides the aperiodicity of that Markov chain and spends a polynomial amount of resources with respect to the size of the input. A formal verification of this solution is presented and a comparative analysis with respect to another known solution is described.

## 1 Introduction

A discrete-time Markov chain is a stochastic process such that the past time is irrelevant to predict the future, given knowledge of the present time. That is, given the present time, the future does not depend on the past time: the result of each event depends only on the result of the previous event.

In order to study the evolution in time of a Markov chain as well as the existence of the stationary distribution, it is suitable to classify its states. This classification depends on the path structure of the chain.

One of the central issues in Markov Theory is the study of the asymptotic behavior of Markov chains. It is well known that for any irreducible and aperiodic Markov chain: (a) there exists at least one stationary distribution (that is, a probability distribution on the state space which is an invariant for the transition matrix associated with the chain), and (b) for any initial distribution, $\mu^{(0)}$ and for any stationary distribution $\pi$ for the Markov chain, the sequence $(\mu^{(n)})_{n\in\mathbf{N}}$ converges to $\pi$ in total variation as $n \to \infty$ (that is, the Markov chain is approaching equilibrium as $n \to \infty$).

In the paper [2], a classification of states of a finite and homogeneous Markov chain is provided by using P systems. Moreover, the period was calculated for recurrent classes. The design of the P systems was inspired in properties used in classic algorithms that deal with the problem of the classification. Especially, this solution allows us to decide whether an irreducible Markov chain is aperiodic or not.

The main goal of this paper is to design a P system associated with an irreducible Markov chain which provides an answer to the aperiodicity of the chain. If the answer is negative, then the system provides the period of the chain. The solution presented is based on a characterization of the aperiodicity in existential terms of some state and a natural number, and it is *semi–uniform*, in the sense that for each Markov chain, a P system associated with it is constructed. Besides, the solution spends a polynomial amount of resources in the sense of the computational complexity theory in Membrane Computing.

The solution presented in the paper improves the solution obtained in [2], because less computational resources are used.

The paper is organized as follows. In the following section, we recall some basic notions and results that we use in the paper. In Section 3, a P system associated with an irreducible Markov chain is designed in order to study the periodicity of that class. Section 4 shows a formal verification of the designed P system. In Section 5, the solution presented is compared with another solution obtained from [2]. Finally some conclusions are presented.

## 2 Preliminaries

A discrete Markov chain is a sequence $\{X_t \mid t \in \mathbb{N}\}$ of random variables whose values are called *states*, that verifies the following property:

$$P(X_{t+1} = j/X_0 = i_0, X_1 = i_1, \ldots, X_t = i_t) = P(X_{t+1} = j/X_t = i_t)$$

Without loss of generality, we can suppose that the state space is the set of nonnegative integers.

The value of variable $X_t$ is interpreted as the state of the process at instant $t$. In this paper we work with Markov chains having a finite state space $S = \{s_1, \ldots, s_k\}$.

A discrete Markov chain is characterized by the *transition probability*

$$p_{ij}(t) = P(X_t = s_j/X_{t-1} = s_i),\ \forall t \geq 1$$

where $p_{ij}(t)$ provides the transition from state $s_i$ to state $s_j$ at time $t-1$.

The matrix of transition probabilities

$$P(t) = (p_{ij}(t))_{1 \le i,j \le k}$$

is a stochastic matrix, that is, is nonnegative for all $t$ and the sum of each arrow is equal to 1, $\sum_{j=1}^{k} p_{ij}(t) = 1$.

We say that the chain is *time homogeneous* or *stationary* if $p_{ij}(t) = p_{ij}$ for each $t$ and it verifies the Kolmogorov-Chapman equation:

$$p_{ij}^{(1)} = p_{ij}, \quad p_{ij}^{(2)} = \sum_{l=1}^{k} p_{il} p_{lj}, \quad \ldots, \quad p_{ij}^{(n)} = \sum_{l=1}^{k} p_{il} p_{lj}^{(n-1)},$$

where $p_{ij}^{(n)}$ is the transition probability of state $s_i$ to state $s_j$ at time $n$.

We denote the initial distribution by means of the vector

$$\mu^{(0)} = (\mu_1^{(0)}, \ldots, \mu_k^{(0)}) = (P(X_0 = s_1), P(X_0 = s_2), \ldots, P(X_0 = s_k))$$

and the distribution of the Markov chain at time $n$ is

$$\mu^{(n)} = (\mu_1^{(n)}, \ldots, \mu_k^{(n)}) = (P(X_n = s_1), P(X_n = s_2), \ldots, P(X_n = s_k))$$

Then, $\mu^{(n)} = \mu^{(0)} \cdot P^{(n)}$, where $P = (p_{ij})$ is the transition matrix of the homogeneous Markov chain.

Next, we introduce some concepts and results related to the states of a homogeneous Markov chain.

We say that a state $s_j$ *communicates* with another state $s_i$ (and we denote it by $s_i \to s_j$), if there exists a natural number $n > 0$ such that $p_{ij}^{(n)} > 0$ (that is, if the chain has a positive probability of ever reaching $s_j$ when we start from $s_i$. We say that the states $s_i$ and $s_j$ *intercommunicate* (and we denote it by $s_i \leftrightarrow s_j$) if $s_i \to s_j$ and $s_j \to s_i$.

In the finite state space $S = \{s_1, \ldots, s_k\}$ of a Markov chain, the relation $\leftrightarrow$ is an equivalence relation and we can consider the corresponding quotient set $\{s_1, \ldots, s_k\}/\leftrightarrow$ whose elements are the classes of equivalence by $\leftrightarrow$.

A Markov chain with state space $S = \{s_1, \ldots, s_k\}$ is said to be *irreducible* if there exists only one class of equivalence by $\leftrightarrow$; that is, if for all $s_i, s_j \in E$ we have $s_i \leftrightarrow s_j$. Otherwise, the chain is said to be *reducible*.

We say that a state $s_i$ is *recurrent* or *essential* if for each natural number $m$ and for each state $s_j$ verifying $p_{ij}^{(m)} > 0$ there exists a natural number $n$ such that $p_{ji}^{(n)} > 0$. Otherwise, the state is said to be *transient*. A recurrent class is the equivalence class determined by a recurrent state.

It is easy to prove that from a recurrent state, only recurrent states belonging to the same class are reachable.

A *recurrence time* of $s_i$ is a natural number $n > 0$ such that $p_{ii}^{(n)} > 0$. The *period* of a state $s_i$ is defined as $d(i) = \text{g.c.d. } \{n \ge 1 \mid p_{ii}^{(n)} > 0\}$, that is, it is the greatest common divisor of the recurrence times associated with it. All states belonging to the same class have the same period.

Then, we can define the period of a class of a given Markov chain in a natural manner: it is the period of any state of the class (see [3] and [4] for more details).

**Definition 1.** *A Markov chain is said to be aperiodic if all its states are aperiodic; that is, their periods are equal to 1. Otherwise, the chain is said to be periodic.*

Next, we provide a method to compute the period of a recurrent class and a characterization of the periodicity of a class.

**Theorem 1.** *Let $A = \{s_1, \ldots, s_r\}$ be a recurrent class. The period of $A$ is*

$$d = g.c.d.\ \{n \mid p_{ii}^{(n)} > 0;\ 1 \leq i, n \leq r\}.$$

*That is, the period of $A$ is the greatest common divisor of all times of recurrences of the states of that class, smaller than or equal to $r$.*

*Proof.* By definition, given a state $s_i$ $(1 \leq i \leq r)$ its period is

$$d(i) = g.c.d.\ \{n \geq 1 \mid p_{ii}^{(n)} > 0\}.$$

As all states have the same period $d$, we have

$$d = d(1) = d(2) = \ldots = d(r) = g.c.d.\ \{n \geq 1 \mid p_{ii}^{(n)} > 0;\ 1 \leq i \leq r\}.$$

Let $d' = g.c.d.\{n \mid p_{ii}^{(n)} > 0;\ 1 \leq i, n \leq r\}$. Let us see that $d = d'$. For that, we will check that any trajectory from a state $s_i \in A$ to itself, with the length bigger than $r$, is the composition of trajectories with length smaller than or equal to $r$ between the same states.

Let $n > r$ be a time of recurrence associated with a state $s_i \in A$, that is, $p_{ii}^{(n)} > 0$. There exists a state $s_{i_0}$ such that $p_{ii}^{(n)} \geq p_{ii_0}^{(n')} \cdot p_{i_0i_0}^{(n_0)} \cdot p_{i_0i}^{(n'')} > 0$, being $n = n' + n_0 + n''$. Thus, $n_0$ and $n' + n''$ are also times of recurrence.

If $n_0 > r$ or $n' + n'' > r$, then we repeat the process until we obtain a decomposition

$$p_{ii}^{(n)} \geq p_{ii_0}^{(n')} \cdot p_{i_0i_0}^{(n_0)} \cdot p_{i_1i_1}^{(n_1)} \ldots p_{i_ri_r}^{(n_r)} \cdot p_{i_ri}^{(n'')} > 0$$

with $1 \leq i_1, \ldots, i_r \leq r$, $n = n' + n_1 + \ldots + n_r + n''$ verifying $n' + n'' \leq r$ and $n_1, \ldots, n_r \leq r$.

Finally, let us notice that substituting $p_{ii}^{(n)}$, with $n > k$, by a *suitable* sequence of $p_{ii}^{(m)}$, with $m \leq k$, the g.c.d. is the same. □

**Lemma 1.** *Let $A = \{a_1, \cdots, a_r\}$ be a set of natural numbers. Let us suppose* g.c.d. $\{a_1, \cdots, a_r\} = 1$. *Let us denote by $A^+$ the set of all positive linear combinations*

$$\lambda_1 a_1 + \cdots, \lambda_r a_r, \quad \textit{with} \quad \lambda_i \in Z^+, 1 \leq i \leq r.$$

*Then, there exists a natural number $N$ such that $n \in A^+$ for all $n \geq N$.*

*Proof.* See, e.g., the appendix of [1] □

Next, we characterize the aperiodicity of a recurrent class of a finite Markov chain through the existence of a state $s_j$ reachable from each state $s_i$.

**Theorem 2.** *Let $\{X_t \mid t \in \mathbb{N}\}$ be a Markov chain with state space $S = \{s_1, \ldots, s_k\}$ and transition matrix $P = (p_{ij})$.*

*(1) If $\{X_t \mid t \in \mathbb{N}\}$ is aperiodic, then there exists a natural number $N$ such that $p_{ii}^{(n)} > 0$, for all $i$ $(1 \leq i \leq k)$ and all $n \geq N$.*
*(2) If $\{X_t \mid t \in \mathbb{N}\}$ is irreducible and aperiodic, then there exists a natural number $M$ such that $p_{ij}^{(n)} > 0$, for all $i, j$ $(1 \leq i, j \leq k)$ and all $n \geq M$.*

*Proof.* See, e.g., Chapter 4 from [3] □

**Theorem 3.** *Let $A = \{s_1, \ldots, s_r\}$ be a recurrent class of a finite Markov chain. The following are equivalent:*

*(1) Class $A$ is aperiodic.*
*(2) There exists a state $s_j \in A$ and a natural number $m_0 \in \mathbb{N}$ such that $p_{ij}^{(m_0)} > 0$ for all state $s_i \in A$.*

*Proof.* Let us suppose that class $A$ is aperiodic. Then all states in $A$ have the same period $d = 1$. From Theorem 2 there exists a natural number $N$ such that $p_{ii}^{(n)} > 0$, for all $i$ $(1 \leq i \leq r)$ and all $n \geq N$. Given $j$ $(1 \leq j \leq r)$, we define $n_i(j) = min\{n \mid p_{ij}^{(n)} > 0\}$, for each $s_i \in A$, $n(j) = max\{n_1(j), \ldots, n_r(j)\}$, and $m_0 = N + n(j)$. Let us see that $p_{ij}^{(m_0)} > 0$, for each $i$ $(1 \leq i \leq r)$. We have $p_{ij}^{(m_0)} \geq p_{ij}^{(n_i(j))} p_{jj}^{(m_0 - n_i(j))} > 0$ because of $p_{ij}^{(n_i(j))} > 0$ by definition of $n_i(j)$, and $p_{jj}^{(m_0 - n_i(j))} > 0$ by Theorem 2.

Conversely, let us suppose that there exists $m_0 \geq 1$ and a state $s_j \in A$ such that $\forall\ s_i \in A$ we have $p_{ij}^{(m_0)} > 0$. In particular, $p_{jj}^{(m_0)} > 0$ so $m_0$ is a recurrence time. On the one hand, if $d$ is the period of the class, then $m_0$ is a multiple of $d$. On the other hand, if $s_i \in A$ is a state such that $p_{ji} > 0$, then $0 < p_{ij}^{(m_0)} p_{ji} \leq p_{ii}^{(m_0+1)}$, so $m_0 + 1$ is a multiple of $d$. Hence, $d = 1$. □

## 3 A P System Associated with an Irreducible Markov Chain

The goal of this paper is to study the aperiodicity of an irreducible Markov chain with state space $S = \{s_1, \ldots, s_k\}$, $k \geq 2$, by using P systems. In the affirmative case, the answer of the system is $YES$, on the contrary, the system sends an object encoding the period of the class to the environment.

### 3.1 The Design of the P System

Let $P_k = (p_{ij})_{1 \leq i,j \leq k}$ be a Boolean matrix associated with a class with a finite and homogeneous Markov chain of order $k$ such that $p_{ij} = 1$ if the transition from $s_i$ to $s_j$ is possible, and $p_{ij} = 0$ otherwise; that is, $P_k$ is the adjacency matrix of the directed graph associated with the recurrent class.

The solution presented in this paper is a *semi–uniform* one in the following sense: we give a family $\mathbf{\Pi} = \{\Pi(P_k) \mid k \in \mathbf{N}\}$, associating with $P_k$ a P system with external output, such that:

- There exists a deterministic Turing machine working in polynomial time which constructs the system $\Pi(P_k)$ from $P_k$.
- The output of the P system $\Pi(P_k)$ provides the classification of the recurrent class of the Markov chain as well as the period of the states.

We associate with the matrix $P_k$ the P system of degree 4 with external output,

$$\Pi(P_k) = (\Gamma(P_k), \mu(P_k), \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, R)$$

defined as follows:

- Working alphabet:

$$\begin{aligned}\Gamma(P_k) = &\{s_{ij},\ t_{ij},\ \tau_{ij} \mid 1 \le i,j \le k\}\ \cup \{s_{ijr} \mid 1 \le i,j,r \le k\} \cup \\ &\{T_r \mid 0 \le r \le k\}\ \cup \{\beta_l \mid 0 \le l \le k-1\} \cup \{b_i \mid 1 \le i \le k\} \cup \\ &\{p_r \mid 1 \le r \le k\}\ \cup\ \{c_i,\ d_i \mid 0 \le i \le \alpha\} \cup \{yes, YES, \sigma, \}\end{aligned}$$

  where $\alpha = 3k + \lceil \frac{k}{2} \rceil$.

  In the working alphabet the objects:
  - $s_{ii}$ represents (at the initial configuration) the state $s_i$ of the chain.
  - $t_{ij}$ and $\tau_{ij}$ represent the elements $p_{ij}$ of the Boolean matrix associated with the transition matrix of the Markov chain.
  - $s_{ijr}$ represents the existence of a path of length $r$ from the state $s_i$ to state $s_j$.
  - $T_r$ and $p_r$ represent the existence of a recurrence time equal to $r$ in different configurations.
  - $\tau_{ij}$ represents that the state $s_j$ is reachable from state $s_i$.
- Membrane structure: $\mu(P_k) = [\ [\ [\ [\ ]_1\ ]_2\ ]_3]_4$.
- Initial multisets:
  $\mathcal{M}_1 = \{t_{ij}^{p_{ij}} \mid 1 \le i,j \le k\} \cup \{\beta_0\}$
  $\mathcal{M}_2 = \{s_{ii} \mid 1 \le i \le k\}$
  $\mathcal{M}_3 = \{b_i \mid 1 \le i \le k\} \cup \{d_0\}$
  $\mathcal{M}_4 = \emptyset$
- The set $R$ of evolution rules consists of the following rules:

$$r_1(ij) \equiv [t_{ij} \to \tau_{ij} t_{ij}^k]_1, \quad 1 \le i,j \le k$$

$$r_2(i) \equiv [\beta_i \to \beta_{i+1}]_1, \quad 0 \le i \le k-2$$

$$r_3 \equiv [\beta_{k-1}]_1 \to c_0^k$$

$$\begin{aligned}r_4(rij) \equiv &[c_r s_{ij} \tau_{j1}^{p_{j1}} \ldots \tau_{jk}^{p_{jk}}]_2 \to [s_{i1}^{p_{j1}} \ldots s_{ik}^{p_{jk}} c_{r+1}^{\gamma_j}]_2 s_{i1r+1}^{p_{j1}} \ldots s_{ikr+1}^{p_{jk}} T_{r+1}^{p_{ji}}, \\ &1 \le i,j \le k, \quad 0 \le r \le \alpha - 1, \gamma_j = \textstyle\sum_{l=1}^{k} p_{jl}\end{aligned}$$

$$r_5 \equiv [\sigma]_2 \rightarrow \sigma$$
$$r_6(jr) \equiv [s_{1jr} \dots s_{kjr}]_3 \rightarrow [\sigma]_2\ yes, \quad 1 \le j \le k,\ 1 \le r \le \alpha$$
$$r_7(r) \equiv [T_r b_r \rightarrow p_r]_3, \quad 1 \le r \le k$$
$$r_8(il) \equiv [p_i p_{i+l} \rightarrow p_i p_l]_3, \quad 1 \le i \le k,\ 1 \le l \le k - i$$
$$r_9(i) \equiv [p_i^2 \rightarrow p_i]_3, \quad 1 \le i \le k$$
$$r_{10}(i) \equiv [d_i \rightarrow d_{i+1}]_3, \quad 0 \le i \le \alpha - 1$$
$$r_{11}(r) \equiv [d_\alpha p_r]_3 \rightarrow p_r[\ ]_3, \quad 2 \le r \le k$$
$$r_{12} \equiv [d_\alpha p_1]_3 \rightarrow yes[\ ]_3$$
$$r_{13} \equiv [yes]_4 \rightarrow YES[\ ]_4$$
$$r_{14}(r) \equiv [p_r]_4 \rightarrow p_r[\ ]_4, \quad 1 \le r \le k$$

### 3.2 An Overview of Computations

Initially, membrane 1 contains objects $t_{ij}$ that codify the elements $p_{ij}$ of the Boolean matrix associated with the transition matrix of the Markov chain, together with the counter $\beta_0$. This counter allows us to dissolve membrane 1 at a certain instant. Membrane 2 contains initially objects $s_{ii}$ that codify the states $s_i$ of the chain. Membrane 3 contains objects $b_i$ that will be used in order to avoid that repeated recurrence times smaller than or equal to $k$ appear. The counter $d$ in membrane 2 will be used to trigger the answer at the suitable instant.

The design of the P system $\Pi(P_k)$ implements a process that is structured by stages. The first one consists of $k$ steps which allow the production of sufficiently many new copies $\tau_{ij}$ of objects $t_{ij}$. This is done by applying rules of type $r_1$ and $r_2$ in membrane 1 at $k-1$ first steps and applying at step $k$ rule $r_3$ that dissolves membrane 1.

At the second stage, all paths between states with length at most $k$, as well as recurrence times smaller than or equal to $k$, are generated. This stage starts at step $k+1$ and it spends at most $k$ steps. First, rules of type $r_4$ are applied producing objects $s_{ijr}$ in membrane 3 that codify the existence of a path with length $r$ from state $s_i$ to state $s_j$, as well as the objects $T_r$ codifying the existence of a recurrence time equal to $r$. Simultaneously, it is checked if there exists a state $s_j$ and a natural number $m_0$ such that $p_{ij}^{(m_0)} > 0$, for all states $s_i$. In that case, an object $\sigma$ is produced in membrane 2 and the system expels an object $YES$ to the environment.

The third stage is only applied if an object $YES$ has not been expelled to the environment. At this stage, the period of the class is computed and it takes $k + \lceil \frac{k}{2} \rceil$ steps. By applying rules of type $r_7$, objects $p_r$ encoding recurrence times smaller than or equal to $k$, are obtained. Such recurrence times are different from each other. By applying rules of types $r_8$ and $r_9$, the greatest common divisor of these times is computed. If the period of the class is equal to 1, then the system sends an object $YES$ to the environment, otherwise, the system expels an object $p_n$ that encodes the period of the class to the environment.

## 4 Formal Verification

Given a computation $\mathcal{C}$ of the P system $\Pi(P_k)$, for each $m \in \mathbf{N}$ we denote by $\mathcal{C}_m$ the configuration of the system obtained after the execution of $m$ steps. For each label $l \in \{1,2,3\}$, we denote by $\mathcal{C}_m(l)$ the multiset of objects contained in membrane $l$ in the configuration $\mathcal{C}_m$. Besides, we denote by $\mathcal{C}_m(env)$ the content of the environment of the system in the configuration $\mathcal{C}_m$.

**Proposition 1.** *(First stage) We have the following:*

*(1) For each $m$, $1 \leq m \leq k-1$, we denote $\psi_m = 1 + k + k^2 + \ldots + k^m = (k^{m+1}-1)/(k-1)$. Then*

$$\mathcal{C}_m(1) = \{\beta_m\ t_{ij}^{k^m \cdot p_{ij}}\ \tau_{ij}^{\psi_{m-1} \cdot p_{ij}}\}.$$

*(2)* $\mathcal{C}_k(2) = \{c_0^k,\ s_{ii},\ \tau_{ij}^{\psi_{k-1} \cdot p_{ij}},\ t_{ij}^{(k^k) \cdot p_{ij}} \mid 1 \leq i,j \leq k\}$

*Proof.* (1) By induction on $m$.

Let us see the result for $m=1$. First, we notice that rule $r_2(0)$ is applicable to configuration $\mathcal{C}_0$, so $\beta_1 \in \mathcal{C}_1(1)$. Rule $r_1(ij)$ is applicable to configuration $\mathcal{C}_0$ if and only if $p_{ij}=1$. Hence, $\mathcal{C}_1(1) = \{\beta_1\ t_{ij}^{k \cdot p_{ij}}\ \tau_{ij}^{p_{ij}}\}$.

Let $m$ be such that $1 \leq m < k-1$ and $\mathcal{C}_m(1) = \{\beta_m\ t_{ij}^{k^m \cdot p_{ij}}\ \tau_{ij}^{\psi_{m-1} \cdot p_{ij}}\}$. Then, rule $r_2(m)$ is applicable to configuration $\mathcal{C}_m$, so $\beta_{m+1} \in \mathcal{C}_{m+1}(1)$. Rule $r_1(ij)$ is applicable to configuration $\mathcal{C}_m$ if and only if $p_{ij}=1$. Hence, $\mathcal{C}_{m+1}(1) = \{\beta_{m+1}\ t_{ij}^{k^m \cdot k \cdot p_{ij}}\ \tau_{ij}^{(\psi_{m-1}+k^m) \cdot p_{ij}}\}$.

(2) From (1), we have $\mathcal{C}_{k-1}(1) = \{\beta_{k-1}\ t_{ij}^{k^{k-1} \cdot p_{ij}}\ \tau_{ij}^{\psi_{k-2} \cdot p_{ij}}\}$. Next, rule $r_1(ij)$ produces $k$ objects $t_{ij}$ and an object $\tau_{ij}$ for each object $t_{ij} \in \mathcal{C}_{k-1}(1)$. Moreover, rule $r_3$ produces $k$ copies of $c_0$ dissolving membrane 1. □

*Remark*: Let us notice that condition $\tau_{ij} \in \mathcal{C}_r(1)$, $1 \leq r \leq k-1$, means that state $s_j$ is reachable from state $s_i$.

**Lemma 2.** *For each $i,j,r$ $(1 \leq i,j,r \leq k)$ we have the following:*

- *The sum of the multiplicities of objects $s_{1j} \ldots s_{kj}$ in $\mathcal{C}_{k+r}(2)$ is, at most, $k^r$.*
- *There exists, at most, $k^{r+1}$ objects $c_r$ in $\mathcal{C}_{k+r}(2)$.*
- $\tau_{ij}^{(\psi_{k-1}-\psi_{r-1}) \cdot p_{ij}} \in \mathcal{C}_{k+r}(2)$.

*Proof.* By induction on $r$.

Let us suppose that $r = 1$, Let $i, j$ be such that $1 \leq i, j \leq k$. From (2) in Proposition 1 we have $\mathcal{C}_k(2) = \{c_0^k,\ s_{ii},\ \tau_{ij}^{\psi_{k-1}\cdot p_{ij}}\} \subseteq \mathcal{C}_k(2)$. Then, rules $r_4(0,1,1), \ldots, r_4(0,k,k)$ must be applied.

If $p_{ij} = 1$, then an object $s_{ij}$ (resp. an object $\tau_{ij}$) is produced (resp. is consumed) by the application of rule $r_4(0,i,i)$ (besides, these objects can only be spent/produced by the application of that rules). Hence, the sum of multiplicities of objects $s_{1j}, \ldots, s_{kj}$ will be $p_{1j} + \ldots + p_{kj} \leq k$, there exists at most $k^2$ objects $c_1$ in $\mathcal{C}_{k+1}(2)$, and $\tau_{ij}^{(\psi_{k-1}-1)} \in \mathcal{C}_{k+1}(2)$.

Let $r \geq 1, r < k$, and let us suppose that the result holds for $r$. Let $i, j$ be such that $1 \leq i, j \leq k$. By the induction hypothesis the sum of the multiplicities of objects $s_{1i}, \ldots, s_{ki}$ in $\mathcal{C}_{k+r}(2)$ is, at most, $k^r$, there exists, at most, $k^{r+1}$ objects $c_r$ in $\mathcal{C}_{k+r}(2)$, and $\tau_{ij}^{(\psi_{k-1}-\psi_{r-1})\cdot p_{ij}} \in \mathcal{C}_{k+r}(2)$. For each $i$ $(1 \leq i \leq k)$ the rules $r_4(r1i), \ldots, r_4(rki)$ will be applied to configuration $\mathcal{C}_{k+r}(2)$ at most $k^r$ times, so at most $k^r$ objects $\tau_{ij}$ will be spent and $k^r \cdot k$ objects $c_{r+1}$ will be produced. Then, there exists at most $k^{r+2}$ objects $c_{r+1}$ in $\mathcal{C}_{k+r+1}(2)$, and $\tau_{ij}^{(\psi_{k-1}-\psi_{r-1}-k^r)\cdot p_{ij}} \in \mathcal{C}_{k+r+1}(2)$.

Moreover, each object $s_{iq}$ $(1 \leq q \leq k)$ produces, at most, an object $s_{ij}$ in $\mathcal{C}_{k+r+1}(2)$. Hence, the sum of multiplicities of $s_{1j}, \ldots, s_{kj}$ in $\mathcal{C}_{k+r+1}(2)$ will be, at most, $k^r + \ldots + k^r = k \cdot k^r = k^{r+1}$. □

**Proposition 2.** *(Second stage) For each $i, j, r$ $(1 \leq i, j, r \leq k)$ we have:*

*(1) Objects $s_{ij}$ and $c_r$ belong to $\mathcal{C}_{k+r}(2)$ if and only if there exists a trajectory from state $s_i$ to state $s_j$ with a length $r$.*

*(2) A state $s$ of the Markov chain has a recurrence time $r$ if and only if $T_r \in \mathcal{C}_{k+r}(3)$.*

*Proof.* (1) By induction on $r$.

Let us suppose that $r = 1$. If $s_{ij}, c_1 \in \mathcal{C}_{k+1}(2)$, then rule $r_4(0,i,i)$ must be applied by using objects $c_0, s_{ii}, \tau_{ij} \in \mathcal{C}_k(2)$. Then, $p_{ij} = 1$, otherwise $p_{ij} = 0 \Rightarrow \tau_{ij} \notin \mathcal{C}_k(2)$ (from Proposition 1).

Let $i_0, j_0$ $(1 \leq i_0, j_0 \leq k)$ and let us suppose that there exists a trajectory from state $s_{i_0}$ to state $s_{j_0}$ with a length 1. Then, $p_{i_0 j_0} = 1$. From Proposition 1 we deduce that $\mathcal{C}_k(2) = \{c_0^k,\ s_{ii},\ \tau_{ij}^{\psi_{k-1}\cdot p_{ij}},\ t_{ij}^{(k^k)\cdot p_{ij}} \mid 1 \leq i, j \leq k\}$, so for each $i$ $(1 \leq i \leq k)$ rule $r_4(0ii)$ is applied once to configuration $\mathcal{C}_k(2)$. Then, $\{c_1, s_{i_0 j_0}\} \subseteq \mathcal{C}_{k+1}(2)$.

Let $r \geq 1$, $r < k$, and let us suppose that the result holds for $r$. Let $i, j$ $(1 \leq i, j \leq k)$ be such that $s_{ij}, c_{r+1} \in \mathcal{C}_{k+r+1}(2)$. On the one hand, rule $r_4(ril)$ has been applied, at least once, to configuration $\mathcal{C}_{k+r}$ by using objects $c_r,\ s_{il},\ \tau_{lj}$ (for some $l$, $1 \leq l \leq k$). So, $p_{lj} = 1$. On the other hand, $c_r,\ s_{il} \in \mathcal{C}_{k+r}(2)$. Then, by induction hypothesis we deduce that there exists a trajectory with a length $r$ from state $s_i$ to state $s_l$. Hence, there exists a trajectory with the length $r + 1$ from state $s_i$ to state $s_j$.

Let $i_0, j_0$ $(1 \leq i_0, j_0 \leq k)$ and let us suppose that there exists a trajectory from state $s_{i_0}$ to state $s_{j_0}$ with a length $r+1$. Then, there exists a trajectory from state $s_{i_0}$ to state $s_{n_0}$ with a length $r$ (for some $n_0$, $1 \leq n_0 \leq k$) such that $p_{n_0 j_0} = 1$. From the induction hypothesis we have $s_{i_0 n_0},\ c_r \in \mathcal{C}_{k+r}(2)$, and from Lemma 2 we deduce that

$$\{\tau_{n_0 j}^{(\psi_{k-1}-\psi_{r-1})\cdot p_{n_0 j}} \mid 1 \leq j \leq k\} \subseteq \mathcal{C}_{k+r}(2)$$

Then, by applying rule $r_4(r, i_0, n_0)$ once, we obtain $\{c_{r+1},\ s_{i_0 j}^{p_{n_0 j}} : 1 \leq j \leq k\} \subseteq \mathcal{C}_{k+r+1}(2)$. Hence, $s_{i_0 j_0} \in \mathcal{C}_{k+r+1}(2)$ because of $p_{n_0 j_0} = 1$.

(2) Let $r$ $(1 \leq r \leq k)$ be the recurrence time of a state $s_i$. From (1), we deduce that $s_{ii}, c_r \in \mathcal{C}_{k+r}(2)$. Therefore, rule $r_4(rij)$ has been applied to configuration $\mathcal{C}_{k+r-1}$, for some $j$, $1 \leq j \leq k$, such that $p_{ji} = 1$, and some object $c_{r-1} \in \mathcal{C}_{k+r-1}(2)$. Then, $T_r^{p_{ji}} = T_r \in \mathcal{C}_{k+r}(3)$.
Let $r$ $(1 \leq r \leq k)$ such that $T_r \in \mathcal{C}_{k+r}(3)$. Then rule $r_4(r-1, i, j)$ has been applied to configuration $\mathcal{C}_{k+r-1}$, for some objects $s_{ij},\ c_{r-1}$ such that $p_{ji} = 1$. From (1) there exists a trajectory with a length $r-1$ from state $s_i$ to state $s_j$. Hence, there exists a trajectory with a length $r$ from state $s_i$ to state $s_i$. □

**Theorem 4.** *(Output of the system)*
*Let $S$ be an irreducible homogeneous Markov chain of order $k$. Let $\alpha = 3k + \lceil \frac{k}{2} \rceil$. We have the following:*

*(1) The class $S$ is aperiodic if and only if there exists $r$ $(1 \leq r \leq \alpha - k)$ such that configuration $\mathcal{C}_{k+r+2}$ of $\Pi(P_k)$ is a halting configuration and $\mathcal{C}_{k+r+2}(env) = \{YES\}$.*
*(2) The class $S$ is periodic with period equal to $n > 1$ if and only if configuration $\mathcal{C}_{\alpha+2}$ of $\Pi(P_k)$ is a halting configuration and $\mathcal{C}_{\alpha+2}(env) = \{p_n\}$.*

*Proof.* Let $S$ be an irreducible homogeneous Markov chain.

(1) Let us suppose that $S$ is aperiodic. From Theorem 3, there exists a state $s_{j_0}$ and a natural number $q > 0$ such that $\forall i$ $(1 \leq i \leq k \Rightarrow p_{ij_0}^{(q)} > 0)$. Then, for each $i$, $1 \leq i \leq k$, there exists a trajectory with a length $q$ from state $s_i$ to state $s_{j_0}$.
- If $q \leq k$, from (1) Proposition 2 we deduce that $s_{1j_0}, \ldots, s_{kj_0}, c_q \in \mathcal{C}_{k+q}(2)$. These objects have been produced by the application of rules $r_4(q-1, 1, j_1), \ldots, r_4(q-1, k, j_k)$ to configuration $\mathcal{C}_{k+q-1}$, for some $j_1, \ldots, j_k$ such that $p_{j_1, j_0} = \cdots = p_{j_k, j_0} = 1$. So, $s_{1j_0q}, \ldots, s_{kj_0q} \in \mathcal{C}_{k+q}(3)$. So, by applying the rule $r_6(j_0, q)$ to configuration $\mathcal{C}_{k+q}$, we have $\{yes\} \in \mathcal{C}_{k+q+1}(4)$ and $\sigma \in \mathcal{C}_{k+q+1}(2)$. At the next step, rules $r_5$ and $r_{13}$ are applied. Then, $\mathcal{C}_{k+q+2}(env) = \{YES\}$, membrane 2 is dissolved and the system halts.
- If $q > k$, then any rule of the type $r_6$ is not applicable. From Proposition 2 we have encoded the recurrence times (smaller than or equal to $k$) in membrane 3 by objects $T$. Then, some rule of the type $r_7$ produces objects $p$ corresponding to objects $T$. Next, by applying suitable rules $r_8$ and $r_9$

we compute the greatest common divisor of these recurrence times (from Theorem 1 we know that g.c.d. is equal to the period of the class). From the aperiodicity of the class $S$, we deduce that object $p_1$ belongs to $\mathcal{C}_\alpha(3)$. Then, the rule $r_{12}$ produces an object *yes* in $\mathcal{C}_{\alpha+1}(4)$, and we obtain that $\mathcal{C}_{\alpha+2}(env) = \{YES\}$ by applying the rule $r_{13}$. Then, the system halts.

Let us suppose that there exists $r$ $(1 \leq r \leq \alpha - k)$ such that configuration $\mathcal{C}_{k+r+2}$ is a halting configuration and $\mathcal{C}_{k+r+2}(env) = \{YES\}$. Then, rule $r_{13}$ has been applied to configuration $\mathcal{C}_{k+r+1}(4)$, and $yes \in \mathcal{C}_{k+r+1}(4)$.

- If $r \leq k$, then the rule $r_6(jr)$ (for some $j$, $1 \leq j \leq k$) has been applied to configuration $\mathcal{C}_{k+r}$, with $s_{1jr}, \dots, s_{kjr} \in \mathcal{C}_{k+r}(3)$, and $s_{1j}, \dots, s_{kj}, c_r \in \mathcal{C}_{k+r}(2)$. From (1) in Proposition 2, there exists a trajectory with a length $r$ from state $s_i$ to state $s_j$, for each $i$ $(1 \leq i \leq k)$. From Theorem 3 we conclude that class $S$ is aperiodic.
- If $r > k$, object *yes* has been sent to membrane 4 by applying the rule $r_{12}$ using objects $d_\alpha$ and $p_1$. But object $p_1$ has been produced by the iterated application of rules $r_8$ and $r_9$. As these rules compute the greatest common divisor of recurrence times, we deduce that the period of $S$ is equal to 1.

(2) Now, let us suppose that the period of $S$ is $n > 1$. From Theorem 3 we deduce that for each $j$, $1 \leq j \leq k$, and for each $n' > 0$ there exists $i$, $1 \leq i \leq k$, such that $p_{ij}^{(n')} = 0$. From (1) in Proposition 2 we have $\{s_{ij}, c_{n'}\} \nsubseteq \mathcal{C}_{k+n'}(2)$. So, $s_{ijn'} \notin \mathcal{C}_{k+n'}(3)$ and any rule of the type $r_6$ is applicable to configuration $\mathcal{C}_{k+n'+1}$. Next, rules of type $r_7, r_8, r_9$ compute the g.c.d. of the recurrence times. Finally, object $p_n$ is sent to the environment after $\alpha + 2$ steps by applying rules $r_{11}$ and $r_{14}$. Then, the system halts.
Let us suppose that configuration $\mathcal{C}_{\alpha+2}$ is a halting configuration and $\mathcal{C}_{\alpha+2}(env) = \{p_n\}$. Then, any rule of the type $r_6$ will not be applied, so rules $r_7$, $r_8$, and $r_9$ will be applied computing the g.c.d of the recurrence times (smaller than or equal to $k$) of states $s_i$. Hence, class $S$ is periodic and its period is equal to $n$. □

## 5 Results and Discussions

In [2] a P system was constructed which allows us to classify the states of a Markov chain. Thus, that P system can be adapted to characterize the aperiodicity of such a chain. Specifically, if $P_k = (p_{ij})_{1 \leq i,j \leq k}$ is the Boolean matrix associated with the states of a recurrent class of a finite and homogeneous Markov chain of order $k$, then we define the system

$$\Pi'(P_k) = (\Gamma'(P_k), \mu'(P_k), \mathcal{M}'_1, \mathcal{M}'_2, \mathcal{M}'_3, \mathcal{M}'_4, R', \rho')$$

as follows:

- Working alphabet:

$$\Gamma'(P_k) = \{d_{ij},\ t_{ij} \mid 1 \le i,j \le k,\} \cup \{c_r \mid 0 \le r \le 2k+2\} \cup \{t_{ijur} \mid 1 \le i,j,u \le k,\ 0 \le r \le k\} \cup \{\beta_i \mid 0 \le i \le \alpha+1\} \cup \{s_{ijr} \mid 1 \le i,j \le k,\ 0 \le r \le k\} \cup \{A_{i1},\ R_{ij} \mid 1 \le i,j \le k\}$$

where $\gamma = 2k + 4 + \lceil \lg_2 k \rceil + \frac{(k-1)(k+2)}{2}$.

- Membrane structure: $\mu'(P_k) = [\ [\ [\ [\ ]_4\ ]_3\ ]_2\ ]_1$.
- Initial multisets:
  $\mathcal{M}'_1 = \emptyset$; $\mathcal{M}'_2 = \{\beta_0\}$; $\mathcal{M}'_3 = \{c_0\}$;
  $\mathcal{M}'_4 = \{s_{ii0} \quad t_{ij}^{p_{ij}(k-1)} \mid 1 \le i,j \le k\}$.
- The set $R$ of evolution rules consists of the following rules:
  - Rules in the skin membrane labeled by 1:
    $r_1 = \{d_{ip} \to (R_{ip}, out) \mid 1 \le i \le k,\ 1 < p \le k\}$
    $r_2 = \{d_{i1} \to (A_{i1}, out) \mid 1 \le i \le k\}$
  - Rules in the membrane labeled by 2:
    $r_3 = \{\beta_i \to \beta_{i+1} \mid 0 \le i \le \gamma\} \cup \{\beta_{\gamma+1} \to \lambda\}$.
    $r_4 = \{d_j^2 \to d_j \mid 1 \le j \le k\}$
    $r_5 = \{d_j d_{j+l} \to d_j d_l \mid 1 \le j \le k,\ 2 \le j+l \le k\}$
  - Rules in the membrane labeled by 3:
    $r_6 = \{t_{ijur} \to (t_{ij} s_{uj(r+1)}, in_4) \mid p_{ij} = 1, u \ne j, 1 \le i,j,u \le k, 0 \le r < 3(k-1)\}$
    $r_7 = \{t_{iju(3k-3)} \to (t_{ij}, in_4) \mid p_{ij} = 1, u \ne j, 1 \le i,j,u \le k\}$
    $r_8 = \{t_{ijjr} \to (t_{ij}, in_4)\ d_{r+1} \mid p_{ij} = 1, 1 \le i,j \le k, 0 \le r < 3(k-1)\}$
    $r_9 = \{t_{ijj(3k-3)} \to (t_{ij}, in_4) \mid p_{ij} = 1, 1 \le i,j \le k\}$
    $r_{10} = \{c_r \to c_{r+1} \mid 0 \le r \le 6(k-1)+1\} \cup \{c_{6(k-1)+2} \to \lambda\}$
  - Rules in the membrane labeled by 4:
    $r_{11} = \{s_{uir} t_{i1}^{p_{i1}} \dots t_{ik}^{p_{ik}} \to (t_{i1ur}^{p_{i1}} \dots t_{ikur}^{p_{ik}}, out) \mid 1 \le u,i \le k,\ 0 \le r \le 3(k-1)\}$.

- The partial order relation $\rho'$ over $R'$ consists of the following relations on the rules of $R'$:
  - Priority relation in the skin membrane: $\emptyset$.
  - Priority relation in the membrane labeled by 2: $\{r_4 > r_5\}$
  - Priority relation in the membranes labeled by 3: $\emptyset$.
  - Priority relation in membrane 4: $\emptyset$.

In order to study the efficiency of the P system $\Pi(P_k)$ constructed in this work, we will compare the results with those obtained by the P system $\Pi'(P_k)$ described above. For that purpose, a comparative analysis of the computational resources required in both P systems is given firstly. Secondly, an analysis of the times of execution obtained on designed simulators for both P systems with some case studies is presented.

### 5.1 Computational Resources Required

The resources required initially to construct the systems $\Pi(P_k)$ and $\Pi'(P_k)$, and the number of steps taken for the systems, are the following:

| | $\Pi(P_k)$ | $\Pi'(P_k)$ |
|---|---|---|
| Size of the alphabet | $\Theta(k^3)$ | $\Theta(k^4)$ |
| Initial number of membranes | 4 | 4 |
| Sum of the sizes of initial multisets | $\Theta(k^2)$ | $\Theta(k^4)$ |
| Number of rules | $\Theta(k^3)$ | $\Theta(k^4)$ |
| Maximal length of a rule | $\Theta(k)$ | $\Theta(k)$ |
| Number of priority relations | 0 | $\Theta(k^2)$ |
| Number of steps | $\Theta(k)$ | $\Theta(k)$ |

In the previous table, let us notice that the amount of resources requested by $\Pi(P_k)$ is smaller than the ones requested by $\Pi'(P_k)$. Indeed, the size of the alphabet and the number of rules pass from power 3 to power 4, and the system $\Pi(P_k)$ has no priority relation. The number of steps is of the same asymptotic order.

### 5.2 Case Studies

We have designed a simulator for each system $\Pi(P_k)$ and $\Pi'(P_k)$. These simulators have been written in C++ language and they have been executed on a Pentium 4 computer with 512 Mb RAM and 3.20 GHz.

In both simulators objects $t_{ij}$ have been represented by means of arrays of dimension 2; objects $s_{ij}$ have been represented by vectors of dimension 2 and recurrent times have been represented by one-dimensional vectors.

The simulator of the system $\Pi(P_k)$ generates the trajectories with a length at most $3k+\lceil k/2\rceil$ in a sequential way, keeping the times of recurrence smaller than or equal to $k$. If assertion (2) in Theorem 3 is fulfilled, the simulator halts displaying the time of execution and the aperiodicity of the Markov chain. Otherwise the simulator computes the g.c.d. of the recurrence times obtained where all of them are different.

Similarly, a simulator for the system $\Pi'(P_k)$ has been implemented. The main difference with respect to the previously mentioned one is that it can keep more than a copy of the times of recurrence. All trajectories of the Markov chain with a length smaller than or equal to $3(k-1)$ and their recurrence time are computed. Then the g.c.d. of these times is obtained.

When the Markov chain is aperiodic, the P system $\Pi(P_k)$ can finish before all trajectories with a length $3k + \lceil k/2\rceil$ are computed. In case it is necessary to calculate the period, bearing in mind that all recurrence times are different, system $\Pi(P_k)$ is faster than $\Pi'(P_k)$ in computing the g.c.d. of these times.

When the Markov chain is periodic the length of the trajectories computed by $\Pi(P_k)$ are longer than those computed by $\Pi'(P_k)$. Nonetheless, in order to compute the period, recurrence times used in $\Pi(P_k)$ are all different.

The simulators designed have been executed on eight recurrent Markov chains with 100 states. Four of these Markov chains are periodic and the others are aperiodic. Table 1 shows the values equal to 1 of the adjacency matrix of the graph associated with the recurrent Markov chains. The execution times are described in Table 2.

| Example | | | |
|---|---|---|---|
| 1 | $p_{i,i+1} = 1$ | $1 \leq i \leq 99$ | |
| | $p_{100,1} = 1$ | | |
| 2 | $p_{i,i+1} = 1$ | $1 \leq i \leq 99$ | |
| | $p_{i,1} = 1$ | $1 \leq i \leq 100$ | |
| 3 | $p_{10j+i,10j+i+1} = 1$ | $1 \leq i \leq 9$ | $0 \leq j \leq 9$ |
| | $p_{10j,10j-9} = 1$ | $1 \leq j \leq 10$ | |
| | $p_{10j+1,10j+11} = 1$ | $0 \leq j \leq 8$ | |
| | $p_{91,1} = 1$ | | |
| 4 | $p_{10j+i,10j+i+1} = 1$ | $1 \leq i \leq 9$ | $0 \leq j \leq 9$ |
| | $p_{10j,10j-9} = 1$ | $1 \leq j \leq 10$ | |
| | $p_{10j+1,10j+11} = 1$ | $0 \leq j \leq 8$ | |
| | $p_{91,1} = 1$ | | |
| | $p_{1,1} = 1$ | | |
| 5 | $p_{10j+i,10j+i+1} = 1$ | $1 \leq i \leq 9$ | $0 \leq j \leq 9$ |
| | $p_{10j,10j-9} = 1$ | $1 \leq j \leq 10$ | |
| | $p_{10j+1,10j+11} = 1$ | $0 \leq j \leq 8$ | |
| | $p_{91,1} = 1$ | | |
| | $p_{2,2} = 1$ | | |
| 6 | $p_{5j+i,5j+i+1} = 1$ | $1 \leq i \leq 4$ | $0 \leq j \leq 19$ |
| | $p_{5j,5j-4} = 1$ | $1 \leq j \leq 20$ | |
| | $p_{5j+1,5j+6} = 1$ | $0 \leq j \leq 18$ | |
| | $p_{96,1} = 1$ | | |
| 7 | $p_{i,i+1} = 1$ | $1 \leq i \leq 100$ | |
| | $p_{i+1,i} = 1$ | $1 \leq i \leq 100$ | |
| | $p_{1+3i,4+3i} = 1$ | $0 \leq i \leq 32$ | |
| 8 | $p_{i,i+1} = 1$ | $1 \leq i \leq 100$ | |
| | $p_{i+1,i} = 1$ | $1 \leq i \leq 100$ | |
| | $p_{1+3i,4+3i} = 1$ | $0 \leq i \leq 32$ | |
| | $p_{1,1} = 1$ | | |

**Table 1.** Adjacency values of the examples

## 6 Conclusions

Markov chains have applications in different fields such as physics, economics, biology, statistics, social sciences. . . In these applications it is important to know whether the Markov chain associated with the process is convergent or not. When the Markov chain is aperiodic, the transition matrix converges and the process becomes stable. In other cases, the process does not reach an equilibrium.

In this work, a characterization of the aperiodicity of a Markov chain has been given in terms of the existence of a state reachable from any other state. Based on this property, a computational P system has been constructed that allows us to know whether the Markov chain is aperiodic and calculate its period if not. A formal verification of P system using the methodology based on the search of invariant formulae has been presented.

| Example | period | Previous | New |
|---|---|---|---|
| 1 | 100 | 0 | 0 |
| 2 | 1 | 146 | 0 |
| 3 | 10 | 0 | 0 |
| 4 | 1 | 122 | 35 |
| 5 | 1 | 1 | 2 |
| 6 | 5 | 11 | 20 |
| 7 | 2 | 381 | 169 |
| 8 | 1 | 1101 | 104 |

**Table 2.** Observed run times

In [2], every finite and homogeneous Markov chain has associated a P system that provides a classification of its recurrent classes. That P system can be adapted to study the aperiodicity of a Markov chain and then its period can be calculated. The solution presented in this work improves the solution derived from the P system described in [2]. For that purpose, simulators have been constructed for these P systems and the respective times of execution on eight examples have been analyzed.

For the computational study of the aperiodicity of a Markov chain it would be interesting to design new P systems that incorporate additional features such as electrical charges, active membranes, etc. and that improve quantitatively the amount of computational resources used.

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